An Explicit Genus-Zero Mirror Principle With Two Marked Points

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Abstract

Mathematical mirror symmetry postulates a relationship between the symplectic geometry of a Calabi-Yau manifold and the complex geometry of a mirror space. The mirror principle program introduced by Lian-Liu-Yau seeks to prove mirror conjectures in a simple and unified framework by exhibiting certain series of equivariant multiplicative characteristic classes on Kontsevich's moduli of stable maps in terms of hypergeometric-type classes. We propose and study an alternate formulation of mirror principle in genus zero that incorporates marked points by using deformations of the Lian-Liu-Yau notion of Euler data. This allows for the explicit construction of good hypergeometric data and establishment of mirror theorems in genus zero in the case of one or two markings by tying computations to the unmarked situation and establishes the foundations for a program involving more markings and higher genus. We discuss the recovery of Gromov-Witten invariants with primary and descendent insertions, and derive several closed formulas, including a one-point version of the famous Candelas formula.

1 Introduction

1.1 Mirror Symmetry

Mirror symmetry arises in physics as a special duality conjectured by supersymmetric string theory. The basic premise of string theory is that particles should be replaced with vibrating strings which propagate through a ten-dimensional universe consisting of the usual four-dimensional space-time $\mathbb{R}^4$ along with a six-dimensional compact Calabi-Yau manifold X [Pol01]. Heuristic physical arguments indicate that there should be an inherent symmetry to the theory on X which interchanges two simultaneous eigenspaces $H^q(X, \Omega^p TX)$ and $H^q(X, \Omega^p_X)$ under the action of supersymmetric charges. As the dimension of the moduli for the theory depends on the dimensions of these eigenspaces, yet these vector spaces are not generally isomorphic for most X, one is led to conjecture the existence of a mirror Calabi-Yau X' such that there is a natural isomorphism $H^q(X, \Omega^p TX) \cong H^q(X', \Omega^p_{X'})$ for all p, q.

Two "topologically twisted" variants of the above model exist, the so-called A-model and B-model. A nice feature of these variants is that the A-model depends only on the symplectic structure of X while the B-model depends only on complex structure. Mirror symmetry may then be phrased conjecturally by saying the A-model on X is equivalent to the B-model on a mirror X' (in fact, often a family of mirror duals). This relationship may be formulated mathematically as the Homological Mirror Symmetry, due to Kontsevich [Kon94], which roughly conjectures an equivalence between the bounded derived category of coherent analytic sheaves on a Calabi-Yau manifold and the derived category of Lagrangian submanifolds in its mirror.

For the purposes of this paper, we will take mirror symmetry to refer to the equality of partition functions, which encode certain measurables of the two topologically twisted theories. The A-model partition function is defined in terms of Gromov-Witten invariants for X. The B-model partition function is determined by the variation of Hodge structure on the mirror space and will either be given in the simplest cases or constructed in this paper.

There are numerous articles written on the origins of mathematical mirror symmetry, for instance [Mor00] and references therein, as well as at least two excellent books [HKK+03] [CK99] introducing the subject.

1.2 Gromov-Witten Theory

Given a smooth projective variety X over $\mathbb{C}$, let $\overline{M}_{g,m}(X, \beta)$ be Kontsevich's moduli space of stable maps to X (see Chapter **??** for definitions and properties). Its points are triples $(f; C; z_1, ..., z_m)$, representing a holomorphic map $f : C \to X$ from a genus g curve with smooth markings $z_1, ..., z_m \in C$ and such that $f_*[C] = \beta \in H_2(X, \mathbb{Z})$, modulo the obvious equivalence. For an excellent introduction to stable maps see [FP96]. The Gromov-Witten theory of X with primary insertions concerns integrals of the form

$$K^X_{g,\beta}(\gamma_1, ..., \gamma_m) = \int_{[\overline{M}_{g,m}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{m} ev_i^* \gamma_i, \tag{1}$$

where integration is over the virtual fundamental class $[\overline{M}_{g,m}(X, \beta)]^{\mathrm{vir}}$ of Li-Tian [LT98] (also Behrend-Fantechi [BF97]), ev_i is the map

$$ev_i : \overline{M}_{g,m}(X, \beta) \to X$$

given by evaluation of f at the i-th marked point, and $\gamma_i \in H^*(X, \mathbb{Z})$. As $\overline{M_{g,m}}(X, \beta)$ is a Deligne-Mumford stack, the Gromov-Witten invariants are in general $\mathbb{Q}$-valued. Philosophically, incidence conditions on X describe cycles on the moduli space, which via an appropriately defined intersection theory lead to invariants approximating interesting enumerative information. Calculation of the invariants (1) and unraveling their connection with enumerative geometry is the fundamental problem in Gromov-Witten theory.

1.3 Mirror Principle

When X is Calabi-Yau, the duality between type IIA and IIB string theories postulates that a generating series for the Gromov-Witten invariants of X (the A-model partition function) is computable in terms of period integrals on the complex moduli of a mirror space (the B-model). This yields, for instance, the mirror conjectures of [CdlOGP91] and [BCOV93] for the quintic threefold, from which one may extract numerous Gromov-Witten invariants in terms of hypergeometric series.

This dissertation will restrict its attention to the mirror principle approach to mirror conjectures due to Bong Lian, Kefeng Liu, and S.-T. Yau [LLY97] [LLY99a] [LLY99b] [LLY01]. The original paper [LLY97], which will be summarized more thoroughly in Chapter 3, develops mirror principle in genus zero without primary insertions for X a projective complete intersection Calabi-Yau or more generally a local Calabi-Yau space arising from a concavex bundle V over $\mathbb{P}^n$ (a direct sum of a positive and negative bundles). Such a bundle $V = V^+ \oplus V^-$ naturally induces a sequence of obstruction bundles V_d on $\overline{M_{0,m}}(\mathbb{P}^n, d)$ whose fiber at $(f; C; z_1, ...z_m)$ is given by $H^0(C, f^*V^+) \oplus H^1(C, f^*V^-)$.

In genus zero, the virtual fundamental class agrees with the usual fundamental class, reducing the calculation of Gromov-Witten invariants (1) for X to enumeration of twisted Euler classes of these obstruction bundles:

$$K_d^X(H^{k_1}, ..., H^{k_m}) = \int_{\overline{M_{0,m}}(\mathbb{P}^n,d)} \text{Euler}(V_d) \prod_{i=1}^{m} ev_i^* H^{k_i} \tag{2}$$

Here H is the hyperplane class on $\mathbb{P}^n$ and we have identified $\beta = d[H]$. We also omit the genus for notational simplicity. The power of this formulation is that a torus T acting on $\mathbb{P}^n$ induces a T-action on $\overline{M_{0,m}}(\mathbb{P}^n, d)$, which permits the localization methods of Atiyah-Bott [AB84] to be applied to an equivariant version of (2).

A central theme in mirror principle is functorial localization (pushing forward localization computations from one T-space to another, stated rigorously in Chapter 2) applied to a complicated nonlinear moduli M_d and its equivariant collapse onto a large projective space N_d. One studies sequences of equivariant classes on N_d called *Euler data* satisfying certain quadratic relations at the fixed points. The spirit is that this gluing should reflect the geometry of fixed components in the nonlinear space.

There are two chief instances of Euler data, corresponding to the A-model and B-model potentials. Lian-Liu-Yau show that an equivariant multiplicative characteristic class of the obstruction bundles V_d induces Euler data. In the case of the Euler class this encodes the Gromov-Witten invariants (2) of interest. Alternatively, certain linear data on N_d having origins in the GKZ-system also will satisfy the quadratic relations. Mirror transformations (invertible maps on the set of Euler data that alter formal cohomology-valued series associated to the data in a simple way) manipulate both of these sequences so a uniqueness theorem applies in the genus-zero unmarked case. One interesting application is a full proof of the famous Candelas formula expressing mirror symmetry for the quintic threefold (also due to Givental [Giv96]).

Much of the machinery in [LLY97] generalizes nicely to cases involving marked points and to higher genus. In particular, Lian-Liu-Yau give a definition for Euler data in such settings and prove a reconstruction algorithm that, in principle at least, leads to all intersection numbers of interest [LLY01]. The difficulty is that the simple linear data corresponding to the B-model was never discovered for comparison.

This dissertation's contribution is a modification to the mirror principle program centered around the philosophy that marked data (and indeed higher genus data) should be recursively determined in terms of unmarked genus-zero data. In the case of one or two marked points, the linear data for comparison is explicitly constructed and pointed mirror theorems are proven (Theorems 6.2 and 6.3).

1.4 Integrality Conjectures in Gromov-Witten Theory

Several integrality conjectures have been advanced connecting the rational numbers produced in (1) with the enumerative geometry of X. When X is a Calabi-Yau threefold, the Aspinwall-Morrison formula [AM93] conjectures integer invariants in genus zero and $m = 0$ after multiple coverings are properly taken into account. This was generalized by Gopakumar and Vafa to a full integratlity conjecture in all genera for Calabi-Yau threefolds [GV]. New integrality conjectures for Calabi-Yau spaces in genus one for dimensions 4 and 5 and an extension in genus zero of the Aspinwall-Morrison formula to n dimensions have also recently been advanced [KP08] [PZ08]. To be specific, in genus zero it is conjectured for Calabi-Yau n-folds that the invariants η uniquely defined via

$$\sum_{\beta\neq 0} K_\beta(\gamma_1, ..., \gamma_m) q^\beta = \sum_{\beta\neq 0} \eta_\beta(\gamma_1, ..., \gamma_m) \sum_{d=1}^{\infty} d^{-3+m} q^{d\beta} \tag{3}$$

are integers.

The author has written a computer program to verify the integrality of these invariants in low degree for a number of cases computable by the methods presented in this dissertation. An example of such integer invariants appears in Section 8.4.

1.5 Outline of Paper

Chapters 2 and 3 comprise background material. Chapter 2 reviews pertinent material on equivariant cohomology, Atiyah-Bott localization, and functorial localization. Chapter 3 provides an overview of [LLY97], which this paper generalizes.

Chapter 4, 5, and 6 constitute the core of the paper. We introduce the notion of (u, v)-Euler data, which can be interpreted as a deformation of Lian-Liu-Yau's Euler data (seen to be the $(0, 0)$ case) and show that a concavex bundle on $\mathbb{P}^n$ indeed induces (u, v)-Euler data. A key uniqueness lemma for the specific case of $(0, 1)$-Euler data is also proved. Chapter 5 gives a natural extension of $(0, 0)$-Euler data to $(0, 1)$-Euler data and discusses certain invertible maps on (u, v)-Euler data known as mirror transformations. The goal is to prove Theorem 5.6, which gives an extension to $(0, 1)$-Euler data with controlled growth in a certain equivariant weight critical to our uniqueness lemma. Chapter 6 combines the results of previous chapters to prove Theorems 6.2 and 6.3, the one-point and two-point mirror theorems which establish the equality of Euler data induced by V with explicitly constructed linear data. When $V = \mathcal{O}(n + 1) \to \mathbb{P}^n$ and working with the equivariant Euler class this agrees with the conclusions of Zinger [Zin07].

Chapter 7 establishes two rather general lemmas for recovering Gromov-Witten invariants from Euler data. In the case of one or two marked points, these can be implemented by computer program to extract Gromov-Witten invariants from Theorems 6.2 and 6.3. An example of this is provided in Chapter 8, along with several closed formulas for Gromov-Witten invariants with field insertions.

2 Localization Techniques

We briefly summarize equivariant cohomology for T-spaces and the powerful technique of localization. For a thorough discussion, the standard reference is [AB84].

2.1 Equivariant Cohomology

Let T be a compact Lie group. There exists a contractible space ET, unique up to homotopy equivalence, on which T acts freely, and with T classified by the principal bundle $ET \to BT$. If X is a topological space with T-action, we define

$$X_T = X \times_T ET,$$

which is a bundle over the classifying space BT with fiber X.

Definition The *equivariant cohomology* of X is defined as

$$H^*_T(X) = H^*(X_T),$$

where $H^*(X_T)$ is the ordinary cohomology of X_T.

We will always take T to be an algebraic torus $(\mathbb{C}^*)^n$, in which case one may check that $BT \cong (\mathbb{P}^\infty)^n$ and $ET = \pi_1^* S \otimes \cdots \otimes \pi_n^* S$, where π_i is projection of BT to the i-th copy of $\mathbb{P}^\infty$ while $S \cong \mathcal{O}(-1)$ is the tautological line bundle on $\mathbb{P}^\infty$. When X is a point we recover the ordinary cohomology of the classifying space BT:

$$H_T^*(pt) = H^*(BT) = \mathbb{C}[\lambda_1, \dots, \lambda_n],$$

where $\lambda_i = c_1(\mathcal{O}(\lambda_i))$ are the weights of the torus action.

Many of the standard concepts in ordinary cohomology translate directly into the equivariant setting, for instance the notions of pullback for T-maps and pushforward by a proper T-map. If we denote inclusion of the fiber by $i_X : X \hookrightarrow X_T$, then in particular the equivariant pullback induces a map $i_X^* : H_T^*(X) \to H^*(X)$ called the *nonequivariant limit*. Furthermore, equivariant pullback by the contraction map $X \to pt$ realizes $H_T^*(X)$ as a $H^*(BT)$-module. The notions of equivariant vector bundle and equivariant characteristic classes are also readily defined.

2.2 Atiyah-Bott Localization

If X be a smooth manifold acted on by a torus T, the T-fixed components of X are known to be a union of smooth submanifolds $\{Z_j\}$ (e.g. [Ive72]). We will denote inclusion of a fixed component into X by $i_{Z_j} : Z_j \hookrightarrow X$ and let the (equivariant) normal bundle of $Z_j \subseteq X$ be denoted by N_{Z_j}. A key observation is that its equivariant Euler class $\mathrm{Euler}_T(N_{Z_j}) \in H_T^*(Z_j)$, which we may pushforward by the Gysin map $i_{Z_j *} : H_T^*(Z_j) \to H_T^*(X)$, is invertible in $H_T^*(X) \otimes \mathcal{R}_T$, where $\mathcal{R}_T \cong \mathbb{C}(\lambda_1, ..., \lambda_n)$ is the field of fractions of $H^*(BT)$.

A fundamental result in the subject is that there is an isomorphism between the equivariant cohomology of X, properly localized, and that of its fixed components

$$H_T^*(X) \otimes \mathcal{R}_T \cong \bigoplus_j H_T^*(Z_j) \otimes \mathcal{R}_T, \tag{4}$$

given explicitly by the map

$$\alpha \to \sum_j \frac{i_{Z_j}^* \alpha}{\mathrm{Euler}_T(N_{Z_j})}.$$

In integrated form,

$$\int_{X_T} \alpha = \sum_{Z_j} \int_{(Z_j)_T} \frac{i_{Z_j}^* \alpha}{\mathrm{Euler}_T(N_{Z_j})}$$

for any $\alpha \in H_T^*(X) \otimes \mathcal{R}_T$.

2.3 Functorial Localization

More generally, let $f : X \to Y$ be a proper equivariant map of T-spaces. If $F \subseteq Y$ is a T-fixed component of Y, $E \subset f^{-1}(F)$ fixed components of $f^{-1}(F)$, and $f_0 := f|_E$, then *functorial localization* [LLY97] states that for $\omega \in H_T^*(X)$,

$$f_{0*}\left(\frac{i_E^*\omega}{\mathrm{Euler}_T(E/X)}\right)=\frac{i_F^*(f_*\omega)}{\mathrm{Euler}_T(F/Y)}.$$

We note that localization techniques extend to hold for smooth spaces supporting a virtual fundamental class [GP99].

3 The Lian-Liu-Yau Mirror Principle

In this section we give an overview of the Lian-Liu-Yau mirror principle program. Ultimately one wishes to show the equivalence of two sequences of equivariant classes on projective spaces that satisfy certain gluing relations on the fixed points. Such sequences are known as Euler data and constitute the primary item of study in mirror principle. We refer the reader to the original paper for further discussion beyond that given here [LLY97].

3.1 Linear and Nonlinear Models

Let $M_d = \overline{M_{0,0}}(\mathbb{P}^n \times \mathbb{P}^1, (d,1))$. This space compactifies the space of all unmarked rational maps of degree d to $\mathbb{P}^n$ and is often called the *nonlinear model* (the map to $\mathbb{P}^n$ is the image of the the map to $\mathbb{P}^n \times \mathbb{P}^1$ specific by a point in M_d, which is invariant under reparametrization of the source curve). M_d has a $G = T \times \mathbb{C}^*$-action, where $T = (\mathbb{C}^*)^{n+1}$, induced by the G-action

$$(t,t_0,\dots,t_n)\cdot([w_0,w_1],[z_0,...,z_n]) = ([t^{-1}w_0,w_1],[t_0^{-1}z_0,\dots,t_n^{-1}z_n]) \qquad (5)$$

on $\mathbb{P}^1 \times \mathbb{P}^n$. Furthermore, there is a natural (equivariant) forgetful morphism π to $\overline{M_{0,0}}(\mathbb{P}^n, d)$ given by forgetting the image on the $\mathbb{P}^1$ factor and stabilizing:

$$\pi : M_d \to \overline{M_{0,0}}(\mathbb{P}^n, d).$$

The G-fixed components of M_d admit a description by decorated graphs as described in Section **??**.

Also define $N_d = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d))^{n+1})$. Since an unmarked degree d rational map to $\mathbb{P}^n$ may be given up to scalar multiple as an $(n+1)$-tuple of degree d homogeneous polynomials in two variables with no common factor, N_d constitutes a different compactification of unmarked rational curves to $\mathbb{P}^n$ of degree d, which we call the *linear model*. It too has a G-action, given by

$$(t,t_0,\dots,t_n)\cdot[\alpha_0(w_0,w_1),\dots,\alpha_n(w_0,w_1)] = [t_0^{-1}\alpha_0(t^{-1}w_0,w_1),\dots,t_n^{-1}\alpha_n(t^{-1}w_0,w_1)]$$

where the α_i are degree d homogeneous polynomials in two variables.

The G-fixed components of N_d are isolated fixed points, which we denote by $\{p_{i,r}\}$ where $0 \leq i \leq n$ and $0 \leq r \leq d$, representing the point $[0, ..., w_1^r w_0^{d-r}, ..., 0]$ with the nonzero term in the i-th slot. We similarly label the T-fixed points of $\mathbb{P}^n$ by $\{q_i\}$ where $0 \leq i \leq n$.

One may compute the equivariant cohomology ring of N_d to be

$$H_G^*(N_d) = \frac{\mathbb{C}[\kappa, \lambda_0, ..., \lambda_n, \hbar]}{\prod_{l=0}^n \prod_{m=1}^d (\kappa - \lambda_l - m\hbar)}$$

where κ is the equivariant hyperplane class, the λ_l are weights of the T-action, and $\hbar$ is the weight of the final $\mathbb{C}^*$-action [CK99]. Let $\mathcal{R}_T H_G^*(N_d)$ denote the localization of $H_G^*(N_d)$ where polynomials in the λ_i are inverted, and $\mathcal{R}_G H_G^*(N_d)$ denote the localization where polynomials in both λ_i and $\hbar$ are inverted.

3.2 Euler Data

Definition A sequence of equivariant classes $Q = \{Q_d\}$ with $Q_d \in \mathcal{R}_T H_G^*(N_d)$ is called Ω-*Euler data* if there is an invertible class $Q_0 = \Omega \in \mathcal{R}_T H_T^*(\mathbb{P}^n)$ so Q satisfy the relations

$$i_{q_i}^*(\Omega) \cdot i_{p_{i,r}}^*(Q_d) = i_{p_{i,r}}^*(Q_r) \cdot i_{p_{i,0}}^*(Q_{d-r})$$

for each $0 \leq i \leq n, d > 0$, and $0 < r < d$.

We will casually refer to the data as Euler data when Ω is understood or unimportant.

Because pullback to $p_{i,r}$ of a class $\omega \in H_G^*(N_d)$ is given by evaluation in the polynomial ring of the equivariant hyperplane class κ at $(\lambda_i + r\hbar)$, it is customary to write $i_{p_{i,r}}^*(\omega)$ as $\omega(\lambda_i + r\hbar)$. This way, a sequence of equivariant classes Q is Ω-Euler data if

$$\Omega(\lambda_i) \cdot Q_d(\lambda_i + r\hbar) = Q_r(\lambda_i + r\hbar) \cdot Q_{d-r}(\lambda_i). \tag{6}$$

We will also need the notion of linked Euler data.

Definition Two Ω-Euler data P and Q are said to be *linked* if $P_d(\lambda_i) = Q_d(\lambda_i)$ at $\hbar = (\lambda_i - \lambda_j)/d$ for all i, $j \neq i$, and d.

Geometrically, Euler data are linked if they agree on multiple coverings of coordinate lines in N_d. The following uniqueness lemma for linked Euler data is proved in [LLY97]:

Lemma 3.1 (Lian-Liu-Yau Uniqueness Lemma) *If Q and P are linked Ω-Euler data such that $deg_\hbar(Q(\lambda_i) - P(\lambda_i)) \leq (n+1)d - 2$ for every $0 \leq i \leq n$ and $d > 0$, then $Q = P$.*

A key result from [LLY97] is the existence of a G-equivariant collapsing map from the nonlinear model to the linear model

$$\varphi : M_d \to N_d. \tag{7}$$

We may take the pullback of any multiplicative equivariant characteristic class b of the bundles V_d on $\overline{M_{0,0}}(\mathbb{P}^n, d)$ via π and then pushforward by the collapsing map φ to obtain a sequence of equivariant classes $\hat{Q} = \{\hat{Q}_d\}$ on the linear model:

$$\hat{Q}_d = \varphi_*(\pi^*(b(V_d))) \in H_G^*(N_d).$$

For most applications, b will be the equivariant Euler class, although the total Chern class is also interesting [LLY97]. It turns out that the sequence $\hat{Q}$ is an instance of Euler data, a fact that will be a special case of the more general Lemma 4.1 proved later in this dissertation. To be more precise, if $V = V^+ \oplus V^-$ is a given concavex bundle, then $\hat{Q}$ induced by V turns out to be $\frac{e(V^+)}{e(V^-)}$-Euler data. Explicitly identifying $\hat{Q}$ in terms of natural linear data having origins in the GKZ-system through use of Lemma 3.1 is the main problem of mirror principle.

4 (u, v)-Euler Data

For any $m \geq 0$, $u + v = m$, and $k_i \geq 0$, let

$$\begin{aligned} \vec{k} &= (k_1, ..., k_m) \\ \vec{k}' &= (k_1, ..., k_v, 0, ..., 0) \\ \vec{k}'' &= (0, ..., 0, k_{v+1}, ..., k_m) \end{aligned} \tag{8}$$

where m, u, v, and the k_i are nonnegative integers. Also let $|\vec{k}| = \sum k_i$.

Definition A $(u + v + 1)$-sequence $Q = \{Q_{d,\vec{k}}\}$ with $Q_{d,\vec{k}} \in \mathcal{R}_T H_G^*(N_d)$ is called (u, v, β, Ω)-*Euler data* if the following hold:

1. $Q_{0,\vec{k}} = \beta^{|\vec{k}|}\Omega$ for some $\beta \in H_T^*(\mathbb{P}^n)$ and all $\vec{k}$.
2. Q satisfies the *Euler condition*

$$\Omega(\lambda_i) \cdot Q_{d,\vec{k}}(\lambda_i + r\hbar) = Q_{r,\vec{k}''}(\lambda_i + r\hbar) \cdot Q_{d-r,\vec{k}'}(\lambda_i). \tag{9}$$

We will call the sequence at some fixed $\vec{k}$ the *height* $\vec{k}$ data and denote it by $Q_{\vec{k}}$. As before, we often suppress β and Ω when unimportant, casually referring to (u, v)-Euler data.

Lian-Liu-Yau's Euler data coincides with $(0,0,\beta,\Omega)$-Euler data. More generally, the height $\vec{0}$ part of any (u,v)-Euler data obeys their gluing relation (6). However, height $\vec{k}\neq\vec{0}$ data is not quite Ω-Euler data nor $\beta^{|\vec{k}|}\Omega$-Euler data but rather a deformation in some sense, and we will say that Q *extends* $Q_{\vec{0}}$. This constitutes a different generalization of Euler data from the one given in [LLY01]. The motivating idea is that we wish to work within a framework that allows one to tie marked data coming from a concavex bundle to unmarked data rather than study the problem for each number of markings independently.

4.1 (u,v)-Euler data and Concavex Bundles

We will now show that a concavex bundle on $\mathbb{P}^n$ naturally induces (u,v)-Euler data when defined through restriction to certain subspaces of an appropriate nonlinear moduli space.

Let $M_d^{u,v}\subset\overline{M_{0,u+v}}(\mathbb{P}^n\times\mathbb{P}^1,(d,1))$ be the subspace of the Deligne-Mumford compactification of maps $\mathbb{P}^1\to\mathbb{P}^n\times\mathbb{P}^1$ of bi-degree $(d,1)$ with $u+v$ marked points such that the first u marked points are mapped to $[0,1]$ under projection to the $\mathbb{P}^1$ factor and the final v marked points are mapped to $[1,0]$ under that projection. We call $M_d^{u,v}$ the *nonlinear model* and again give it a G-action induced by multiplication on the image as in (5). Note that $M_d^{0,0}$ is the nonlinear moduli M_d used by Lian-Liu-Yau. The natural projection to $\overline{M_{0,u+v}}(\mathbb{P}^n,d)$ will again be denoted by π, suppressing the u,v dependence:

$$\pi: M_d^{u,v}\longrightarrow\overline{M_{0,u+v}}(\mathbb{P}^n,d)$$

As in the unmarked case, there exists an equivariant collapsing map to N_d, still to be denoted $\varphi: M_d^{u,v}\to N_d$, defined as composition of the Lian-Liu-Yau collapsing map (7) from $\overline{M_{0,0}}(\mathbb{P}^n\times\mathbb{P}^1,(d,1))$ to N_d with the forgetful morphism(s) from $M_d^{u,v}$ to $\overline{M_{0,0}}(\mathbb{P}^n\times\mathbb{P}^1,(d,1))$ forgetting the markings.

Fix $u+v=m$ and again work with the sequence of obstruction bundles V_d induced by a concavex bundle $V=V^+\oplus V^-$. For $\vec{k}=(k_1,...,k_m)$, let

$$\chi_{d,\vec{k}}^V=\pi^*\left(b(V_d)\prod_{j=1}^m ev_j^*(p^{k_j})\right)\in H_G^*(M_d^{u,v}),$$

where p is the equivariant hyperplane class on $\mathbb{P}^n$ and b is any multiplicative equivariant characteristic class. Also let $\hat{Q}=\{\hat{Q}_{d,\vec{k}}\}$ be given as

$$\hat{Q}_{d,\vec{k}}=\varphi_*(\chi_{d,\vec{k}}^V)\in H_G^*(N_d).$$

The $m=0$ case is precisely the Euler data $\hat{Q}$ studied in [LLY97]. Our immediate goal is to prove the following:

Lemma 4.1 *$\hat{Q}$ is (u, v, p, Ω)-Euler data, where $\Omega = \frac{e(V^+)}{e(V^-)}$ and p is the equivariant hyperplane class on $\mathbb{P}^n$.*

We first give some setup and prove a crucial gluing lemma (Lemma 4.2). Although the obstruction bundles V_d on $\overline{M_{0,m}}(\mathbb{P}^n, d)$ exist on different spaces, there are non-trivial relations on multiplicative equivariant characteristic classes connecting them. For any $0 < r < d$, define $\overline{M}^{i,j}(r, d-r)$ by the pullback diagram

$$\begin{array}{ccc} \overline{M}^{i,j}(r, d-r) & \xrightarrow{p_1} & \overline{M_{0,i+1}}(\mathbb{P}^n, r) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle ev_{i+1}} \\ \overline{M_{0,j+1}}(\mathbb{P}^n, d-r) & \xrightarrow{ev_{j+1}} & \mathbb{P}^n \end{array}$$

where p_1 and p_2 are projections and ev_k evaluation at the k-th marked point. We think of $\overline{M}^{i,j}(r, d-r)$ as consisting of pairs of an $(i+1)$-pointed stable map of degree r and a $(j+1)$-pointed stable map of degree $d-r$, $((f_1, C_1, w_1, ..., w_{i+1}), (f_2, C_2, z_1, ..., z_{j+1}))$, such that $f_1(w_{i+1}) = f_2(z_{j+1}) = q$. We also have the diagram

$$\begin{array}{ccc} \overline{M}^{i,j}(r, d-r) & \xrightarrow{\psi} & \overline{M_{0,i+j}}(\mathbb{P}^n, d) \\ \downarrow{\scriptstyle ev} & & \\ \mathbb{P}^n & & \end{array}$$

where the image of ev is q, and the image of ψ is the stabilization of the map formed by connecting C_1 at w_{i+1} with C_2 at z_{j+1} of degree d given by $f|_{C_k} = f_k$.

In the case that $r = 0$ and $i = 0$ or 1, $\overline{M_{0,i+1}}(\mathbb{P}^n, r)$ is empty. To remedy this we replace $\overline{M_{0,i+1}}(\mathbb{P}^n, 0)$ with a copy of $\mathbb{P}^n$, and replace ev_{i+1} with $id_{\mathbb{P}^n}$. This way, $\overline{M}^{i,j}(0, d) \cong \overline{M_{0,j+1}}(\mathbb{P}^n, d)$. We let ψ forget the $(j+1)$-th point when $i = 0$ and be the identity when $i = 1$. This way ψ results in an $(i+j)$-point curve in all situations. The case $r = d$ is similarly modified when $j = 0$ or 1.

Lemma 4.2 (Gluing Lemma) *If b is a multiplicative equivariant characteristic class then the following relation on $\overline{M}^{i,j}(r, d-r)$ holds for $0 < r < d$:*

$$ev^*\Omega \cdot \psi^* b(V_d) = p_1^* ev_{i+1}^* b(V_r) \cdot p_2^* ev_{j+1}^* b(V_{d-r})$$

where $\Omega = \frac{b(V^+)}{b(V^-)}$.

Proof Suppose $((f_1, C_1, w_1, ..., w_{i+1}), (f_2, C_2, z_1, ..., z_{j+1}))$ map to $(f, C, w_1, ..., w_i, z_1, ..., z_j)$ under ψ. This gives rise to the exact sequence

$$0 \to f^*V \to f_1^*V \oplus f_2^*V \to V|_q \to 0$$

In the case that V is convex, passing to cohomology gives the exact sequence

$$0 \to H^0(C, f^*V) \to H^0(f_1^*V) \oplus H^0(f_2^*V) \to V|_q \to 0,$$

from which the lemma follows easily. Likewise, if V is concave we have the exact sequence

$$0 \to V|_q \to H^1(C, f^*V) \to H^1(f_1^*V) \oplus H^1(f_2^*V) \to 0,$$

and again the corresponding gluing lemma follows. The general concavex case is then an easy combination of the two cases. ∎

We now prove Lemma 4.1:

Proof First observe that $\hat{Q}_{d,\vec{k}}$ is automatically polynomial in $\hbar$, as it is an equivariant class on N_d. We show that $\hat{Q}$ satisfies (9) by pulling the calculation back to $M_d^{u,v}$:

$$\hat{Q}_{d,\vec{k}}(\lambda_i + r\hbar) = \int_{(N_d)_G} \phi_{p_{i,r}} \hat{Q}_{d,\vec{k}} = \int_{(M_d^{u,v})_G} \varphi^*(\phi_{p_{i,r}}) \chi^V_{d,\vec{k}}, \tag{10}$$

where $\phi_{p_{i,r}} = \prod_{(j,s)\neq(i,r)}(\kappa - (\lambda_j + s\hbar))$ is the equivariant Poincaré dual to the fixed point $p_{i,r}$. We wish to calculate (10) via localization and so must understand the G-fixed components. Each fixed component $M_\Gamma \subseteq M_d$ is labeled by a decorated graph Γ in the usual way [CK99] [HKK+03], although we will not directly need this description. The only components contributing are those such that $\varphi(M_\Gamma) = p_{i,r}$.

We first consider the case $0 < r < d$. To obtain a better description of these G-fixed components and in particular their normal bundles inside of $M_d^{u,v}$, we build a variation $\overline{\psi}$ of the map ψ given above as follows: for each point of $\overline{M}^{u,v}(r, d-r)$ represented by the pair $((f_1, C_1, w_1, ..., w_{u+1}), (f_2, C_2, z_1, ..., z_{v+1}))$ with $f_1(w_{u+1}) = q = f_2(z_{v+1})$, let $C = C_0 \cup C_1 \cup C_2$ where $C_0 \cong \mathbb{P}^1$, with $[1, 0] \in C_0$ glued to $w_{u+1} \in C_1$, and $[0, 1] \in C_0$ glued to $z_{v+1} \in C_2$. Then define $f : C \to \mathbb{P}^1 \times \mathbb{P}^n$ by

$$\begin{aligned} f|_{C_0}(z) &= (z, q) \\ f|_{C_1}(z) &= ([1, 0], f_1(z)) \\ f|_{C_2}(z) &= ([0, 1], f_2(z)) \end{aligned}$$

This induces a map $\overline{M}^{u,v}(r, d-r) \to \overline{M_{0,u+v}}(\mathbb{P}^n \times \mathbb{P}^1, (d, 1))$. Moreover, the image is contained in $M_d^{u,v}$ and so defines a map $\overline{\psi}$

$$\bar{\psi} : \overline{M}^{u,v}(r, d-r) \hookrightarrow M_d^{u,v}.$$

Fix $0 \leq i \leq n$ for the moment and let $\{F_r^j\}$ denote the set of T-fixed components of $\overline{M_{0,j}}(\mathbb{P}^n, r)$ such that the j-th marked point is sent to $q_i \in \mathbb{P}^n$. In the present case, it is clear that $F_r^{u+1} \times F_{d-r}^{v+1} \subseteq \overline{M}^{u,v}(r, d-r)$, and we identify this subset with its image under $\bar{\psi}$, writing $F_r^{u+1} \times F_{d-r}^{v+1} \subseteq M_d^{u,v}$. It is not hard to see that all the fixed components M_Γ of $M_d^{u,v}$ with $\varphi(M_\Gamma) = p_{i,r}$ arise in this way. Hence by functorial localization, (10) becomes

$$\hat{Q}_{d,\vec{k}}(\lambda_i + r\hbar) = \sum_{\{F_r^{u+1} \times F_{d-r}^{v+1}\}} \int_{(F_r^{u+1} \times F_{d-r}^{v+1})_G} \frac{i_\Gamma^*(\varphi^*(\phi_{p_{i,r}})\chi_{d,\vec{k}}^V)}{\mathrm{Euler}_G(N_{F_r^{u+1} \times F_{d-r}^{v+1}})}.$$

We want to identify the class $\mathrm{Euler}_G(N(F_r^{u+1} \times F_{d-r}^{v+1}))$ in the equivariant K-theory. By considering the various inclusions at hand it is straightforward to find

$$N(F_r^{u+1} \times F_{d-r}^{v+1}) = N(F_r^{u+1}) + N(F_{d-r}^{v+1}) - 2T_{q_i}\mathbb{P}^n + T_{q_i}\mathbb{P}^n + N(\bar{\psi}) - N(J),$$

where J is the inclusion map $J : M_d^{u,v} \hookrightarrow \overline{M_{0,u+v}}(\mathbb{P}^n \times \mathbb{P}, (d, 1))$. Thus, omitting pullbacks for simplicity,

$$\mathrm{Euler}_G(N(F_r^{u+1} \times F_{d-r}^{v+1})) = \mathrm{Euler}_G(N(\bar{\psi}))\frac{\mathrm{Euler}_T(N(F_r^{u+1}))\mathrm{Euler}_T(N(F_{d-r}^{v+1}))}{\mathrm{Euler}_G(N(J))\mathrm{Euler}_T(T_{q_i}\mathbb{P}^n)}.$$

These pieces may be identified as

$$\mathrm{Euler}_T(T_{q_i}\mathbb{P}^n) = \prod_{j \neq i}(\lambda_i - \lambda_j)$$
$$\mathrm{Euler}_G(N(J))) = (-1)^v \hbar^{u+v}$$
$$\mathrm{Euler}_G(N(\bar{\psi})) = -\hbar^2(-\hbar - c_1^G(\mathcal{L}_{d-r,v+1}))(\hbar - c_1^G(\mathcal{L}_{r,u+1}))$$

where $\mathcal{L}_{r,j}$ is the line bundle on $\overline{M_{0,j}}(\mathbb{P}^n, d)$ given by the cotangent line to the j-th marked point on each curve. Furthermore, by Lemma 4.2, on $F_r^{u+1} \times F_{d-r}^{v+1}$

$$\Omega(\lambda_i)b(V_d) = b(V_r)b(V_{d-r})$$

Putting all this together, we have:

$$\begin{aligned}\hat{Q}_{d,\vec{k}}(\lambda_i + r\hbar) &= \int_{(M_d^{u,v})_G} \varphi^*(\phi_{i,r})\chi^V_{d,\vec{k}} \\ &= (-1)^v \hbar^{u+v-2}(\Omega(\lambda_i))^{-1} i^*_{p_{i,r}}(\phi_{i,r}) \prod_{j\neq i}(\lambda_i - \lambda_j)\times \\ &\quad \sum_{\{F_r^{u+1}\}} \int_{(F_r^{u+1})_T} \frac{\pi^*_{u+1}(b(V_r)\prod_{j=1}^u ev_j^* p^{k_{j+v}})}{\mathrm{Euler}_T(N(F_r^{u+1}))(\hbar - c_1^G(\mathcal{L}_{r,u+1}))}\times \\ &\quad \sum_{\{F_{d-r}^{v+1}\}} \int_{(F_{d-r}^{v+1})_T} \frac{\pi^*_{v+1}(b(V_{d-r})\prod_{j=1}^v ev_j^* p^{k_j})}{\mathrm{Euler}_T(N(F_{d-r}^{v+1}))(\hbar + c_1^G(\mathcal{L}_{d-r,v+1}))}. \qquad (11)\end{aligned}$$

A similar analysis for $r = 0$ yields

$$\hat{Q}_{d,\vec{k}}(\lambda_i) = (-1)^v i^*_{p_{i,0}}(\phi_{i,0})\hbar^{v-1} \prod_{j=1}^{u} \lambda_i^{k_{v+j}} \sum_{\{F_d^{v+1}\}} \int_{(F_d^{v+1})_T} \frac{\pi^*_{v+1}(b(V_d)\prod_{j=1}^v ev_j^* p^{k_j})}{\mathrm{Euler}_G(N(F_d^{v+1}))(\hbar + c_1^G(\mathcal{L}_{d,v+1}))} \qquad (12)$$

so that

$$\hat{Q}_{d-r,\vec{k}'}(\lambda_i) = (-1)^v i^*_{p_{i,0}}(\phi_{i,0})\hbar^{v-1} \sum_{\{F_{d-r}^{v+1}\}} \int_{(F_{d-r}^{v+1})_T} \frac{\pi^*_{v+1}(b(V_{d-r})\prod_{j=1}^v ev_j^* p^{k_j})}{\mathrm{Euler}_G(N(F_{d-r}^{v+1}))(\hbar + c_1^G(\mathcal{L}_{d-r,v+1}))}. \qquad (13)$$

Likewise, for $r = d$ we find

$$\hat{Q}_{r,\vec{k}''}(\lambda_i + r\hbar) = i^*_{p_{i,r}}(\phi_{i,r})\hbar^{u-1} \sum_{\{F_r^{u+1}\}} \int_{(F_r^{u+1})_T} \frac{\pi^*_{u+1}(b(V_r)\prod_{j=1}^u ev_j^* p^{k_{j+v}})}{\mathrm{Euler}_G(N(F_r^{u+1}))(\hbar - c_1^G(\mathcal{L}_{r,u+1}))}, \qquad (14)$$

Combining (11), (13), and (14), it is then straightforward to check that that $\hat{Q}$ satisfies (9) and thus is (u, v, p, Ω)-Euler data. (Note that in (11), $i^*_{p_{i,r}}(\phi_{i,r})$ is on N_d but in (13) $i^*_{p_{i,0}}(\phi_{i,0})$ is on N_r and in (14) $i^*_{p_{i,r}}(\phi_{i,r})$ on N_{d-r}.) ∎

4.2 Uniqueness for (0,1)-Euler data

In this section we will restrict ourselves to studying a uniqueness lemma for $(0, 1)$-Euler data. The discussion has a natural analog for $(1, 0)$-Euler data, which we will not state, that will only be used briefly in the discussion of two markings.

For notational simplicity, we will refer to the height using k rather than $\vec{k} = (k_1)$.

Lemma 4.3 (Uniqueness Lemma) *If $Q = \{Q_{d,k}\}$ and $P = \{P_{d,k}\}$ are $(0, 1, \beta, \Omega)$-Euler data such that $Q_0 = P_0$ and $deg_\hbar(Q_{d,k}(\lambda_i) - P_{d,k}(\lambda_i)) \leq (n+1)d - 1$ for all i, k, and d, then $Q = P$.*

Proof We will prove $Q_{d,k} = P_{d,k}$ for each k by induction on d. Notice that $Q_{0,k} = \beta^k \Omega = P_{0,k}$ by assumption. Suppose that for any fixed $k > 0$, $Q_{d',k} = P_{d',k}$ for $0 < d' < d$. Since the intersection pairing is nondegenerate, it will be sufficient to show that

$$\int_{(N_d)_G} \kappa^s (Q_{d,k} - P_{d,k}) = 0 \qquad \text{for all } s \geq 0,$$

where κ is the equivariant hyperplane class on N_d. We compute this integral by localization:

$$\begin{aligned}
\int_{(N_d)_G} \kappa^s (Q_{d,k} - P_{d,k}) &= \sum_{0 \leq i \leq n} \sum_{0 \leq r \leq d} \frac{(\lambda_i + r\hbar)^s i^*_{p_{i,r}} (Q_{d,k} - P_{d,k})}{\mathrm{Euler}_G(N_{p_{i,r}})} \\
&= \sum_{i=0}^{n} \left[\frac{\lambda_i^s (Q_{d,k}(\lambda_i) - P_{d,k}(\lambda_i))}{\mathrm{Euler}_G(N_{p_{i,0}})} + \frac{(\lambda_i + d\hbar)^s (Q_{d,k}(\lambda_i + d\hbar) - P_{d,k}(\lambda_i + d\hbar))}{\mathrm{Euler}_G(N_{p_{i,d}})} \right] \\
&= \sum_{i=0}^{n} \frac{\lambda_i^s (Q_{d,k}(\lambda_i) - P_{d,k}(\lambda_i))}{\mathrm{Euler}_G(N_{p_{i,0}})}
\end{aligned}$$

The second line follows from the first because Q satisfies the Euler condition, allowing $Q_{d,k}(\lambda_i + r\hbar)$ to be expressed in terms of $Q_{1,k}, \ldots, Q_{d-1,k}$ and $Q_{r,0}$ whenever $0 < r < d$, and likewise for $P_{d,k}(\lambda_i + r\hbar)$. By the inductive hypothesis these agree, and the only terms remaining in the difference are the $r = 0$ and $r = d$ cases. Furthermore, using the assumption $Q_0 = P_0$, when $r = d$

$$\begin{aligned}
Q_{d,k}(\lambda_i + d\hbar) &= \Omega(\lambda_i)^{-1} Q_{d,0}(\lambda_i + d\hbar) Q_{0,k}(\lambda_i) \\
&= \Omega(\lambda_i)^{-1} P_{d,0}(\lambda_i + d\hbar) P_{0,k}(\lambda_i) \\
&= P_{d,k}(\lambda_i + d\hbar),
\end{aligned}$$

so this difference vanishes as well giving the final line. Computing the normal bundle in the final line, we find that

$$\int_{(N_d)_G} \kappa^s (Q_{d,k} - P_{d,k}) = \sum_{i=0}^{n} \frac{\lambda_i^s A_i(\hbar)}{\hbar^d} \tag{15}$$

where

$$A_i(\hbar) = \frac{(-1)^d}{d!\prod_{j\neq i}(\lambda_i-\lambda_j)}\frac{Q_{d,k}(\lambda_i)-P_{d,k}(\lambda_i)}{\prod_{j\neq i}\prod_{m=1}^{d}(\lambda_i-(\lambda_j+m\hbar))}.$$

We claim that A_i is in fact polynomial in $\hbar$. It is easy to see that $(0,1,\beta,\Omega)$-Euler data automatically satisfies the *self-linking condition* that $Q_{d,k}(\lambda_i) = \beta(\lambda_j)^k Q_{d,0}(\lambda_i)$ at $\hbar = (\lambda_i - \lambda_j)/d$ for all i, $j \neq i$, k, and d. It follows that at $\hbar = (\lambda_i - \lambda_j)/d$

$$Q_{d,k}(\lambda_i) - P_{d,k}(\lambda_i) = \beta(\lambda_j)^k(Q_{d,0}(\lambda_i) - P_{d,0}(\lambda_i)) = 0,$$

which cancels out some zeros of the denominator. The other zeros at $\hbar = (\lambda_i - \lambda_j)/s$ where $0 < s < d$ cancel as well again by repeated use of the self-linking condition (geometrically if $Q_{d,k}$ and $Q_{s,k}$ are both multiples of $Q_{d,0}$ on covers of coordinate lines then they are multiples of each other). We have shown that A_i is indeed a polynomial in $\hbar$. However, the left side of (15) is naturally a polynomial in $\hbar$. The degree bound implies that

$$\deg_\hbar A_i \leq (n+1)d - 1 - nd = d - 1$$

Since $\hbar^d$ divides A_i we conclude that A_i, and hence the left hand side of (15), always vanish. The lemma then follows. ∎

5 Hypergeometric Data and Mirror Transformations

Throughout this section and the next, we will take $\beta \in H_T^*(\mathbb{P}^n)$ to be p, the equivariant hyperplane class. The goal is first to extend given $(0,0)$-Euler data to $(0,1)$-Euler data in a natural way, and then to transform $(0,1)$-Euler data into "linked" data so certain hypergeometric series associated to the data obey simple relations. These notions will be made precise shortly. Ultimately we prove Theorem 5.6 showing that this extension can be made in a way that controls the growth of $\hbar$.

5.1 An Extension Lemma

Let $\mathcal{S}_0$ denote the set of all sequences $B = \{B_d\}$ such that $B_d \in H_T^*(\mathbb{P}^n)(\hbar)$ for all d, where $\hbar$ is now a formal parameter. The hypergeometric function associated to $B \in \mathcal{S}_0$ is a formal function taking values in $H_T^*(\mathbb{P}^n) \otimes \mathbb{C}(\hbar)$ given by

$$HG[B](t) = e^{-pt/\hbar}\left[B_0 + \sum_{d=1}^{\infty} e^{dt}\frac{B_d}{\prod_{l=0}^{n}\prod_{m=1}^{d}(p-\lambda_l-m\hbar)}\right].$$

Let $I_d : \mathbb{P}^n \hookrightarrow N_d$ be the equivariant embedding given by

$$[a_0, \ldots, a_n] \mapsto [a_0 w_1^d, \ldots, a_n w_1^d]. \tag{16}$$

Also let $\mathcal{S}$ denote the set of all sequences $Q = \{Q_d\}$ of equivariant classes with $Q_d \in \mathcal{R}_T H_G^*(N_d)$. We then have a natural map

$$\mathcal{I} : \mathcal{S} \to \mathcal{S}_0,$$

defined by $\mathcal{I}(Q)_d = I_d^*(Q_d)$. With this notation, the hypergeometric series associated to the height $\vec{k}$ sequence in (u, v, p, Ω)-Euler data is given by

$$HG[\mathcal{I}(Q_{\vec{k}})](t) = e^{-pt/\hbar} \left[\Omega p^{|\vec{k}|} + \sum_{d=1}^{\infty} e^{dt} \frac{I_d^*(Q_{d,\vec{k}})}{\prod_{l=0}^{n} \prod_{m=1}^{d} (p - \lambda_l - m\hbar)} \right]. \tag{17}$$

For the remainder of this section we will deal only with $(0, 1, p, \Omega)$-Euler data, and again for simplicity adopt the notation Q_k in place of $Q_{\vec{k}}$ for the data at a particular height.

Lemma 5.1 (Extension Lemma) *There is a map* $\rho : \mathcal{S} \to \mathcal{S}$ *such that* ρ *gives a natural extension of* $(0, 0, -, \Omega)$*-Euler data* $Q_0 \in \mathcal{S}$ *to* $(0, 1, p, \Omega)$*-Euler data* Q *in such a way that*

$$Q_k = \rho^k(Q_0)$$

for all $k > 0$ *and with the property that*

$$HG[\mathcal{I}(Q_k)](t) = \left\{ -\hbar \frac{\partial}{\partial t} \right\}^k HG[\mathcal{I}(Q_0)](t). \tag{18}$$

Proof Given $Q_0 = \{Q_{0,k}\}$, define $\rho^k(Q_0) = \{Q_{d,k}\}$ via

$$Q_{d,k}(\lambda_i + r\hbar) = (\lambda_i - (d - r)\hbar)^k Q_{d,0}(\lambda_i + r\hbar).$$

That this extension satisfies 9 is easily checked, and as the restriction at each fixed point of N_d is polynomial in $\hbar$ so is the class $Q_{d,k}$ itself. Furthermore,

$$i_{q_i}^* I_d^*(Q_{d,k}) = Q_{d,k}(\lambda_i) = (\lambda_i - d\hbar)^k Q_{d,0}(\lambda_i)$$

for every fixed point $q_i \in \mathbb{P}^n$, from which we conclude

$$I_d^*(Q_{d,k}) = (p - d\hbar)^k I_d^*(Q_{d,0}).$$

Equation (18) is now verified by straightforward power series manipulation. ∎

5.2 The Lagrange Lift

In this section we introduce the Lagrange lift, which maps collections of equivariant classes on $\mathbb{P}^n$ to collections of equivariant classes on N_d, and can be thought of a sort of inverse to the map $\mathcal{I}$ previously introduced. We also state specific conditions under which this lifted data is (u, v)-Euler data.

To be specific, fix $u, v \geq 0$, and let $\vec{k}$, $\vec{k'}$ and $\vec{k''}$ be defined as in (8). Also fix an invertible equivariant class $\Omega \in H_T^*(\mathbb{P}^n)$.

Definition The *Lagrange lift* of a collection of sequences $B = \{B_{d,\vec{k}}\}$ indexed by $\vec{k}$ and with $B_{d,\vec{k}} \in \mathcal{R}_T H^*(\mathbb{P}^n) \otimes \mathbb{C}(\hbar)$ is the collection of sequences $\mathcal{L}(B) = \{\mathcal{L}(B)_{d,\vec{k}}\}$ defined (via localization (4)) by

$$\mathcal{L}(B)_{d,\vec{k}}(\lambda_i + r\hbar) = \Omega(\lambda_i)^{-1}\overline{B_{r,\vec{k''}}(\lambda_i)}B_{d-r,\vec{k'}}(\lambda_i) \tag{19}$$

where $0 \leq r \leq d$ and $0 \leq i \leq n$, so that $\mathcal{L}(B)_{d,\vec{k}} \in \mathcal{R}_G H_G^*(N_d)$.

Here overline denotes the conjugate map on $\mathbb{P}^n$ sending $\hbar$ to $-\hbar$ (and more generally on N_d) mentioned in the introduction and studied thoroughly in [LLY97].

Lemma 5.2 *If $B = \{B_{d,\vec{k}}\}$ is a collection of sequences indexed by $\vec{k}$ where $B_{d,\vec{k}} \in \mathcal{R}_T H^*(\mathbb{P}^n) \otimes \mathbb{C}(\hbar)$, then $\mathcal{L}(B)$ is (u, v, p, Ω)-Euler data if and only if*

1. *$B_{0,\vec{k}} = p^{|\vec{k}|}\Omega$, and*
2. *$\mathcal{L}(B)_{d,\vec{k}}$ is polynomial in $\hbar$.*

Moreover, if $\mathcal{L}(B)$ is (u, v)-Euler data, then $B_{d,\vec{k}} = p^{|\vec{k''}|}B_{d,\vec{k'}}$.

Proof Suppose the lift $\mathcal{L}(B)$ is (u, v, p, Ω)-Euler data. $\mathcal{L}(B)_{d,\vec{k}}$ is then polynomial in $\hbar$ by definition. Since $I_d \circ i_{q_i} = i_{p_{i,0}}$,

$$B_{0,\vec{k}}(\lambda_i) = \mathcal{L}(B)_{0,\vec{k}}(\lambda_i) = p^{|\vec{k}|}(\lambda_i)\Omega(\lambda_i)$$

for all $0 \leq i \leq n$, giving $B_{0,\vec{k}} = p^{|\vec{k}|}\Omega$.

Now suppose the three conditions in (5.2) hold. The lift in (19) is designed so lifted data $\mathcal{L}(B)$ satisfies the Euler condition (9), while the third condition immediately implies $\mathcal{L}(B)_{d,\vec{k}} \in \mathcal{R}_T H_G^*(N_d)$. To check that $\mathcal{L}(B)_{0,\vec{k}} = p^{|\vec{k}|}\Omega$, observe that

$$\begin{aligned}
\mathcal{L}(B)_{0,\vec{k}}(\lambda_i) &= \Omega(\lambda_i)^{-1}B_{0,\vec{k''}}(\lambda_i)B_{0,\vec{k'}}(\lambda_i) \\
&= \Omega(\lambda_i)^{-1}p^{|\vec{k''}|}(\lambda_i)\Omega(\lambda_i)p^{|\vec{k'}|}(\lambda_i)\Omega(\lambda_i) \\
&= p^{|\vec{k}|}(\lambda_i)\Omega(\lambda_i)
\end{aligned}$$

at each fixed point.

For the final statement, observe

$$\begin{aligned}\mathcal{L}(B)_{d,\vec{k}}(\lambda_i) &= \Omega(\lambda_i)^{-1}\mathcal{L}(B)_{0,\vec{k}''}\mathcal{L}(B)_{d,\vec{k}'}(\lambda_i)\\ &= \Omega(\lambda_i)^{-1}\Omega(\lambda_i)p^{|\vec{k}''|}(\lambda_i)\Omega(\lambda_i)^{-1}\Omega(\lambda_i)B_{d,\vec{k}'}(\lambda_i)\\ &= p^{|\vec{k}''|}(\lambda_i)B_{d,\vec{k}'}(\lambda_i),\end{aligned}$$

from which one concludes $B_{d,\vec{k}} = p^{|\vec{k}''|}B_{d,\vec{k}'}$. ∎

One nice property of the Lagrange lift is that it serves as a sort of inverse to the map $\mathcal{I}$, a fact that we will use freely. To show this, we calculate that on collections of data B satisfying the two hypotheses of Lemma 5.2,

$$\begin{aligned}\mathcal{I}(\mathcal{L}(B)_{d,\vec{k}})(\lambda_i) &= \mathcal{L}(B)_{d,\vec{k}}(\lambda_i)\\ &= \Omega(\lambda_i)^{-1}\overline{B_{0,\vec{k}''}}(\lambda_i)B_{d,\vec{k}'}(\lambda_i) && \text{(by 19)}\\ &= \Omega(\lambda_i)^{-1}p^{|\vec{k}''|}B_{0,\vec{0}}(\lambda_i)B_{d,\vec{k}'}(\lambda_i) && \text{(by Lemma 5.2)}\\ &= p^{|\vec{k}''|}B_{d,\vec{k}'}(\lambda_i) && \text{(Lemma 5.2 again)}\\ &= B_{d,\vec{k}}(\lambda_i).\end{aligned}$$

Since the localization to each fixed point agrees, the classes are equal. This shows $\mathcal{I}\circ\mathcal{L} = \mathrm{id}$ on such data.

Likewise, when restricting to collections of (u, v, p, Ω)-Euler data Q, we have

$$\begin{aligned}\mathcal{L}(\mathcal{I}(Q_{d,\vec{k}}))(\lambda_i + r\hbar) &= \Omega(\lambda_i)^{-1}\overline{\mathcal{I}(Q_{r,\vec{k}''})}(\lambda_i)\mathcal{I}(Q_{d-r,\vec{k}'})(\lambda_i)\\ &= \Omega(\lambda_i)^{-1}\overline{Q_{r,\vec{k}''}}(\lambda_i)Q_{d-r,\vec{k}'}(\lambda_i)\\ &= Q_{d,\vec{k}}(\lambda_i + r\hbar).\end{aligned}$$

This establishes that $\mathcal{L}\circ\mathcal{I} = \mathrm{id}$ when restricted to (u, v, p, Ω)-Euler data.

We want to establish a practical way to check the second condition in Lemma 5.2, that $\mathcal{L}(B)_{d,\vec{k}}$ is polynomial in $\hbar$. Suppose B is data satisfying the hypotheses of Lemma 5.2; multiplying both sides of (19) by the identity

$$\begin{aligned}e^{d\tau}\frac{e^{(\lambda_i+r\hbar)(t-\tau)/\hbar}}{\prod_{j=0}^{n}\prod_{m=0,(j,m)\neq(i,r)}^{d}(\lambda_i + r\hbar - \lambda_j - m\hbar)} =& \frac{1}{\prod_{j\neq i}(\lambda_i - \lambda_j)} \times \frac{e^{rt}}{\prod_{j=0}^{n}\prod_{m=1}^{r}(\lambda_i - \lambda_j + m\hbar)}\\ &\times e^{\lambda_i(t-\tau)/\hbar}\frac{e^{(d-r)\tau}}{\prod_{j=0}^{n}\prod_{m=1}^{r}(\lambda_i - \lambda_j - m\hbar)}\end{aligned}$$

and summing over $0 \leq i \leq n$, $0 \leq r \leq d$, and then finally over $d = 0$ to ∞ yields (via localization) the identity

$$\sum_{d \geq 0} e^{d\tau} \int_{(N_d)_G} e^{\kappa(t-\tau)/\hbar} \mathcal{L}(B)_{d,\vec{k}} = \int_{(\mathbb{P}^n)_T} \Omega^{-1} \overline{HG[B_{\vec{k}''}](t)} HG[B_{\vec{k}'}](\tau) \tag{20}$$

Equation (20) is generally useful in establishing that lifts under the Lagrange map are polynomial in $\hbar$ and will be used to establish the existence of mirror transformations in the next section.

5.3 Mirror Transformations

By Lemma 5.1, initial $(0, 0, \Omega)$-Euler data may be extended to $(0, 1, p, \Omega)$-Euler data in a simple way. However, the hypergeometric data associated to this extension will behave badly in that it grows in powers of $\hbar$ as the height k increases. This uncontrolled growth disallows the use of the uniqueness result for $(0, 1)$-Euler data, Lemma 4.3. To control the hypergeometric data of the extension we will show the existence of certain *mirror transformations*, which transform $(0, 1)$-Euler data into new *linked* $(0, 1)$-Euler data. More generally, mirror transformations may be used to manipulate (u, v)-Euler data.

We now introduce some natural extensions to definitions found in [LLY97]:

Definition Two (u, v, β, Ω)-Euler data P and Q are called *linked* if Q_0 and P_0 are linked $(0, 0, \Omega)$-Euler data, meaning that $Q_{d,0}(\lambda_i)$ and $P_{d,0}(\lambda_i)$ agree at $\hbar = (\lambda_i - \lambda_j)/d$ for all i, j, and d.

Notice that if (u, v)-Euler data Q and P are linked, then by virtue of (9), $Q_{d,\vec{k}}(\lambda_i)$ and $P_{d,\vec{k}}(\lambda_i)$ agree at $\hbar = (\lambda_i - \lambda_j)/d$ for all $\vec{k}$. Let $\mathcal{A}^\Omega$ denote the set of all (u, v, p, Ω)-Euler data.

Definition A *mirror transformation* on (u, v, β, Ω)-Euler data is an invertible map $\mu : \mathcal{A}^\Omega \to \mathcal{A}^\Omega$ so that $\mu(Q)$ is linked to Q for all $Q \in \mathcal{A}^\Omega$. We will sometimes deal with mirror transformations that alter the Euler data at one particular height $\vec{k}$ while fixing all other heights; in this case, we call μ a *mirror transformation of height* $\vec{k}$. Clearly the composition of mirror transformations of different heights will be a mirror transformation.

Lemmas 5.3, 5.4, and 5.5 below describe specific mirror transformations on (u, v, p, Ω)-Euler data that will be needed to establish Theorem 5.6 or in the examples. The existence of a mirror transformation described at each height by a relation on hypergeometric data is established by first showing that there is a well-defined collection of data in $\mathcal{S}_0$ satisfying the relation, that this data lifts well to a data in $\mathcal{S}$, and that moreover the lifted data is indeed Euler data by showing it satisfies the conditions of Lemma 5.2.

If one ignores the higher height data, Lemmas 5.3 and 5.4 exactly reduce to Lemma 2.15 of [LLY97]. Their proofs are modifications of that argument, which we shall produce here for completion.

Lemma 5.3 *There exists a mirror transformation* $\mu : \mathcal{A}^\Omega \to \mathcal{A}^\Omega$ *so that for any* $g \in e^t \mathcal{R}_T[[e^t]]$,

$$HG[\mathcal{I}(\mu(Q)_{\vec{k}})](t) = HG[\mathcal{I}(Q_{\vec{k}})](t+g)$$

for all $Q \in \mathcal{A}^\Omega$ *and every height* $\vec{k}$.

Proof Let $\mathcal{I}(Q_{\vec{k}}) = B_{\vec{k}} \in \mathcal{S}_0$, so that

$$HG[B_{\vec{k}}](t+g) = e^{-pt/\hbar} e^{-pg/\hbar} \sum_{d=0}^{\infty} e^{dt} e^{dg} \frac{B_{d,\vec{k}}}{\prod_{l=0}^{n} \prod_{m=1}^{d} (p - \lambda_l - m\hbar)}.$$

Expand

$$e^{dg} = 1 + \sum_{s \geq 1} g_{d,s} e^{st} \qquad g_{d,s} \in \mathcal{R}_T$$
$$e^{-pg/\hbar} = 1 + \sum_{s \geq 1} g'_s e^{st} \qquad g'_s \in \mathcal{R}_T[p/\hbar]$$

It is straightforward to find that setting

$$\tilde{B}_{d,\vec{k}} = B'_{d,\vec{k}} + \sum_{r=0}^{d-1} g'_{d-r} B'_{r,\vec{k}} \prod_{l=0}^{n} \prod_{m=r+1}^{d} (p - \lambda_l - m\hbar),$$

where

$$B'_{d,\vec{k}} = B_{d,\vec{k}} + \sum_{r=0}^{d-1} g_{r,d-r} B_{r,\vec{k}} \prod_{l=0}^{n} \prod_{m=r+1}^{d} (p - \lambda_l - m\hbar),$$

uniquely defines $\tilde{B}_{\vec{k}} = \{\tilde{B}_{d,\vec{k}}\}$ for each $\vec{k}$ in such a way that

$$HG[\tilde{B}_{\vec{k}}](t) = HG[B_{\vec{k}}](t+g).$$

Notice that $\tilde{B}_{0,\vec{k}} = B_{0,\vec{k}}$. By Lemma 5.2, we may now define $\tilde{Q} = \mathcal{L}(\tilde{B})$ provided we check that $\mathcal{L}(\tilde{B})_{d,\vec{k}} \in \mathcal{R}_T H^*_G(N_d)$. Using 20,

$$\sum_{d\geq 0} e^{d\tau} \int_{(N_d)_G} e^{\kappa(t-\tau)/\hbar}\tilde{Q}_{d,\vec{k}}$$
$$= \int_{(\mathbb{P}^n)_T} \Omega^{-1}\overline{HG[\mathcal{I}(\mu(Q)_{\vec{k}''})](t)}HG[\mathcal{I}(\mu(Q)_{\vec{k}'})](\tau)$$
$$= \int_{(\mathbb{P}^n)_T} \Omega^{-1}\overline{HG[\mathcal{I}(Q_{\vec{k}''})](t+g(e^t))}HG[\mathcal{I}(Q_{\vec{k}'})](\tau+g(e^\tau))$$
$$= \sum_{d\geq 0} e^{d\tau} \int_{(N_d)_G} e^{\kappa(t+\bar{g}(e^t)-\tau-g(e^\tau))/\hbar}Q_{d,\vec{k}}$$

Decompose $g = g_+ + g_-$, where $\bar{g}_\pm = \pm g_\pm$, and set $q = e^\tau$ and $\zeta = (t-\tau)/\hbar$, so that $g_+(qe^{\zeta\hbar}) - g_+(q) \in \hbar\mathcal{R}_T[[q,\zeta]]$. Moreover, $g_-(q)$, $g_-(qe^{\zeta\hbar}) \in \hbar\mathcal{R}_T[[q,\zeta]]$ naturally. Since $Q_{d,\vec{k}} \in \mathcal{R}_T H^*_G(N_d)$, the final summation above lies in $\mathcal{R}_T[[q,\zeta]]$, and so too must the initial summation. It follows that

$$\int_{(N_d)_G} \kappa^s \tilde{Q}_{d,\vec{k}} \in \mathcal{R}_T$$

for all $s \geq 0$. A priori $\tilde{Q}_{d,\vec{k}} = a_N\kappa^N + ... + a_0$, with $a_i \in \mathcal{R}_G$, and pairing this expression with various powers of κ and integrating over $(N_d)_G$ gives $a_i \in \mathcal{R}_T$. Thus $\tilde{Q}_{d,\vec{k}} \in \mathcal{R}_T H^*_G(N_d)$ is (u,v)-Euler data and we may define the mirror transformation μ via $\mu(Q) = \tilde{Q}$. ∎

Lemma 5.4 *There exists a mirror transformation* $\nu : \mathcal{A}^\Omega \to \mathcal{A}^\Omega$ *of height* $\vec{l}$ *such that for any* $f \in \hbar e^t \mathcal{R}_T[[e^t]]$,

$$HG[\mathcal{I}(\nu(Q)_{\vec{l}})](t) = e^{f/\hbar} HG[\mathcal{I}(Q_{\vec{l}})](t).$$

If $\vec{l} = \vec{0}$*,* f *maybe be taken to be in* $e^t\mathcal{R}_T[[e^t]]$.

Proof We are given (u,v)-Euler data $Q = \{Q_{d,\vec{k}}\}$ and a specific height $\vec{l}$ at which we wish to alter the hypergeometric data. Let $B = \{B_{d,\vec{k}}\}$ be defined via $B_{\vec{k}} = \mathcal{I}(Q_{\vec{k}})$. Decompose f via $e^{f/\hbar} = 1 + \sum_{s\geq 1} f_s e^{st}$, where $f_s \in \mathcal{R}_T$ when $\vec{l} \neq \vec{0}$ or $f_s \in \mathcal{R}_G$ in the $\vec{l} = \vec{0}$ case of the lemma. It is straightforward to check that the collection of data $\tilde{B}$ defined by

$$\tilde{B}_{d,\vec{l}} = B_{d,\vec{l}} + \sum_{r=0}^{d-1} f_{d-r} B_{r,\vec{l}} \prod_{j=0}^{n} \prod_{m=r+1}^{d} (p - \lambda_j - m\hbar)$$

at height $\vec{l}$ and $\tilde{B}_{d,\vec{k}} = B_{d,\vec{k}}$ at all other heights $\vec{k} \neq \vec{l}$ satisfies

$$HG[\tilde{B}_{\vec{l}}](t) = e^{f/\hbar} HG[B_{\vec{l}}](t)$$

while the hypergeometric data at all other heights is of course unaltered.

We want to lift this new collection $\tilde{B}$. First observe that $\tilde{B}_{0,\vec{k}} = B_{0,\vec{k}}$ at *all* heights, which is the first condition in Lemma 5.2. Lemma 5.2 may then be used to conclude that $\mathcal{L}(\tilde{B})$ is (u,v)-Euler data, provided that we check $\mathcal{L}(\tilde{B})_{d,\vec{k}}$ is always polynomial in $\hbar$.

Let $\tilde{Q} = \mathcal{L}(\tilde{B})$. By (20),

$$\begin{aligned} &\sum_{d\geq 0} e^{d\tau} \int_{(N_d)_G} e^{\kappa(t-\tau)/\hbar} \tilde{Q}_{d,\vec{l}} \\ &\qquad = \int_{(\mathbb{P}^n)_T} \Omega^{-1} \overline{HG[\mathcal{I}(\tilde{Q}_{\vec{l}''})](t)} HG[\mathcal{I}(\tilde{Q}_{\vec{l}'})](\tau) \end{aligned} \tag{21}$$

We will investigate the various cases for $\vec{l}$.

Case 1 If $\vec{l} \neq \vec{l}'$ or $\vec{l}''$, the hypergeometric data in (21) is unaltered, and it is immediate that $\tilde{Q}_{d,\vec{l}}$ is polynomial in $\hbar$.

Case 2 If $\vec{l} = \vec{0}$, then $\vec{l}', \vec{l}'' = \vec{0}$. Both hypergeometric series in (21) are transformed, and the calculation continues as

$$\begin{aligned} &\int_{(\mathbb{P}^n)_T} \Omega^{-1} \overline{HG[\mathcal{I}(\tilde{Q}_{\vec{0}})](t)} HG[\mathcal{I}(\tilde{Q}_{\vec{0}})](\tau) \\ &\qquad = \int_{(\mathbb{P}^n)_T} \Omega^{-1} \overline{e^{f(t)/\hbar} HG[\mathcal{I}(\mathcal{L}(B)_{\vec{0}})](t)} e^{f(\tau)/\hbar} HG[\mathcal{I}(\mathcal{L}(B)_{\vec{0}})](\tau) \\ &\qquad = \int_{(\mathbb{P}^n)_T} \Omega^{-1} e^{-(f_+(t)-f_-(t))/\hbar} e^{(f_+(\tau)+f_-(\tau))/\hbar} \overline{HG[B_{\vec{0}}](t)} HG[B_{\vec{0}}](\tau) \\ &\qquad = \sum_{d\geq 0} e^{d\tau} e^{-(f_+(t)-f_+(\tau))/\hbar} e^{(f_-(t)+f_-(\tau))/\hbar} \int_{(N_d)_G} e^{\kappa(t-\tau)/\hbar} Q_{d,\vec{0}}, \end{aligned}$$

where $\overline{f_\pm} = \pm f_\pm$. Recall we are allowing $f \in \mathcal{R}_T[[e^t]]$ at this height. An argument similar to the one given at the end of Lemma 5.3 shows that $\tilde{Q}_{d,\vec{0}}$ is polynomial in $\hbar$. It follows that there exists a mirror transformation on (u,v)-Euler data of height $\vec{0}$ which alters the hypergeometric data by multiplication by $e^{f/\hbar}$.

Case 3 We now consider the scenario in which $\vec{l} = \vec{l}'$ but $\vec{l} \neq \vec{l}''$. The reverse situation is a modification in notation and will not be given.

By (20),

$$\sum_{d\geq 0} e^{d\tau}\int_{(N_d)_G} e^{\kappa(t-\tau)/\hbar}\tilde{Q}_{d,\vec{l}}$$
$$= \int_{(\mathbb{P}^n)_T} \Omega^{-1}\overline{HG[\mathcal{I}(\tilde{Q}_{\vec{l}''})](t)}HG[\mathcal{I}(\tilde{Q}_{\vec{l}'})](\tau)$$
$$= \int_{(\mathbb{P}^n)_T} \Omega^{-1}HG[\mathcal{I}(\mathcal{L}(B)_{\vec{l}''})](t)e^{f(\tau)/\hbar}HG[\mathcal{I}(\mathcal{L}(B)_{\vec{l}'})](\tau)$$
$$= \sum_{d\geq 0} e^{d\tau}e^{f(\tau)/\hbar}\int_{(N_d)_G} e^{\kappa(t-\tau)/\hbar}Q_{d,\vec{l}}.$$

Set $q = e^{\tau}$ and $\zeta = (t-\tau)/\hbar$. Since $\vec{l}\neq\vec{0}$ we have assumed $f(\tau)\in\hbar q\mathcal{R}_T[[q]]$. The final line then lies in $\mathcal{R}_T[[q,\zeta]]$ because Q is Euler data. It easily follows that $\tilde{Q}_{d,\vec{l}}$ is polynomial in $\hbar$ since $Q_{d,\vec{l}}$ is polynomial in $\hbar$. ∎

The previous two lemmas describe mirror transformations that generalize those in [LLY97] used to manipulate the the $\hbar^{-1}$-expansion of hypergeometric data so a uniqueness result applies. A new class of mirror transformations that allows one to further manipulate the degree 0 term of the $\hbar^{-1}$-expansion of higher derivatives of hypergeometric data in such a way as to eliminate all the purely equivariant terms that appear after applying Lemma 5.1 will also be necessary. The following lemma provides that new class of transformations:

Lemma 5.5 *For $(0,v)$-Euler data, there exists a mirror transformation $\eta : \mathcal{A}^{\Omega}\to\mathcal{A}^{\Omega}$ of arbitrary height $\vec{l}\neq\vec{0}$ such that for any $f\in e^t\mathcal{R}_T[[e^t]]$ and any $\vec{m}$,*

$$HG[\mathcal{I}(\eta(Q)_{\vec{l}})](t) = HG[\mathcal{I}(Q_{\vec{l}})](t) + f\cdot HG[\mathcal{I}(Q_{\vec{m}})](t)$$

for all $Q\in\mathcal{A}^{\Omega}$.

Proof Let the collection of data $B=\{B_{d,\vec{k}}\}$ be defined by $B_{\vec{k}}=\mathcal{I}(Q_{\vec{k}})$. We need to establish that there is a collection of data $\tilde{B}=\{\tilde{B}_{d,\vec{k}}\}$ such that

$$HG[\tilde{B}_{\vec{l}}](t) = HG[B_{\vec{l}}](t) + f(t)\cdot HG[B_{\vec{m}}](t),$$

while $HG[\tilde{B}_{\vec{k}}](t) = HG[B_{\vec{k}}](t)$ for all $\vec{k}\neq\vec{l}$, and that this collection lifts well under the Lagrange map to $(0,v)$-Euler data linked to Q.

It is straightforward to calculate that under the decomposition $f(t)=\sum_{s\geq 1} f_s e^{st}$, where $f_s\in\mathcal{R}_T$, the desired collection $\tilde{B}$ is given uniquely by

$$\tilde{B}_{d,\vec{l}} = B_{d,\vec{l}} + \sum_{s=1}^{d} f_s B_{d-s,\vec{m}}\prod_{l=0}^{n}\prod_{m=d-s+1}^{d}(p-\lambda_l-m\hbar)$$

at height $\vec{l}$ and $\tilde{B}_{d,\vec{k}} = B_{d,\vec{k}}$ otherwise. We observe that $\tilde{B}_{0,\vec{k}} = B_{0,\vec{k}}$ at all heights $\vec{k}$. Let $\tilde{Q} = \mathcal{L}(\tilde{B})$. We need to check that that $\tilde{Q}_{d,\vec{k}}$ is in fact in $\mathcal{R}_T H^*_G(N_d)$. This is immediate except in the case $\vec{k} = \vec{l}$. Since this is $(0, v)$-Euler data, $\vec{l} = \vec{l}'$ and $\vec{l}'' = \vec{0}$, and likewise for $\vec{m}$. Applying (20), we have

$$\begin{aligned}
&\sum_{d\geq 0} e^{d\tau} \int_{(N_d)_G} e^{\kappa(t-\tau)/\hbar} \tilde{Q}_{d,\vec{l}} \\
&\quad = \int_{(\mathbb{P}^n)_T} \Omega^{-1} \overline{HG[\tilde{B}_{\vec{l}''})](t)} HG[\tilde{B}_{\vec{l}'})](\tau) \\
&\quad = \int_{(\mathbb{P}^n)_T} \Omega^{-1} \overline{HG[\mathcal{I}(Q_{\vec{0}})](t)} \left(HG[\mathcal{I}(Q_{\vec{l}})](\tau) + f(\tau) \cdot HG[\mathcal{I}(Q_{\vec{m}'})](\tau) \right) \\
&\quad = \sum_{d\geq 0} e^{d\tau} \int_{(N_d)_G} e^{\kappa(t-\tau)/\hbar} \left[Q_{d,\vec{l}} + f(\tau) Q_{d,\vec{m}} \right]
\end{aligned}$$

Both $Q_{d,\vec{k}}$ and $Q_{d,\vec{k}'}$ are in $\mathcal{R}_T H^*_G(N_d)$, as Q is Euler data. An argument similar to that at the end of the proof of Lemma 5.3 shows that $\tilde{Q}_{d,\vec{k}}$ must also be in $\mathcal{R}_T H^*_G(N_d)$. The desired mirror transformation η is then the map sending Q to $\tilde{Q}$. ∎

The lemma could also be stated for $(u, 0)$-Euler data, although we will not have a need to explicitly use this.

5.4 A Controlled Extension Lemma

We now have all the ingredients necessary to prove an extension lemma for $(0, 1)$-Euler data that controls the growth of $\hbar$. We again simplify notation by denoting the height for $(0, 1)$-data by k rather than $\vec{k} = (k)$.

Theorem 5.6 *Suppose that P_0 is $(0, 0, \Omega)$-Euler data whose hypergeometric data has series expansion in $\hbar^{-1}$ of the form*

$$HG[\mathcal{I}(P_0)](t) = \Omega \left[1 + \sum_{q=1}^{\infty} \sum_{r=0}^{q} (-1)^q y^r_{0,q}(t) p^{q-r} \hbar^{-q} \right],$$

where $y^r_{0,q}(t)$ are degree-r symmetric polynomials in $\{\lambda_i\}$. Then P_0 may be extended to $(0, 1, p, \Omega)$-Euler data P in such a way that the hypergeometric data at each height k has series expansion of the form

$$HG[\mathcal{I}(P_k)](t) = \Omega p^k \left[1 + \sum_{q=1}^{\infty} \sum_{r=0}^{q+k} (-1)^q y^r_{k,q}(t) p^{q-r} \hbar^{-q} \right],$$

where $y^r_{k,q}(t)$ are degree-r symmetric polynomials in $\{\lambda_i\}$ and the $y^0_{k,q}(t)$ are recursively defined by

$$y^0_{k,q}(t) = \frac{{y^0_{k-1,q+1}}'(t)}{{y^0_{k-1,1}}'(t)}$$

for $1 \leq k \leq q$.

Proof We construct the $(0,1)$-Euler data P by induction on height. The existence of height 0 data satisfying the theorem is presupposed. Suppose that for all $0 \leq k' \leq k$, $P_{k'}$ has been constructed with hypergeometric data

$$HG[\mathcal{I}(P_{k'})](t) = \Omega p^{k'} \left[1 + \sum_{q=1}^{\infty} \sum_{r=0}^{q+k'} (-1)^q y^r_{k',q}(t) p^{q-r} \hbar^{-q} \right].$$

We wish to construct the height $k+1$ data. The operator $\rho : \mathcal{S} \to \mathcal{S}$ from Lemma 5.1 acting on P_k gives height $k+1$ data $\rho(P_k)$ such that

$$\begin{aligned} HG[\mathcal{I}(\rho(P_k))](t) &= -\hbar \frac{\partial}{\partial t} \left(HG[\mathcal{I}(P_k)](t) \right) \\ &= \Omega p^k \left[\sum_{r=0}^{k+1} {y^r_{k,1}}'(t) p^{1-r} + \sum_{q=2}^{\infty} \sum_{r=0}^{q+k} (-1)^{q+1} {y^r_{k,q}}'(t) p^{q-r} \hbar^{-q+1} \right]. \end{aligned}$$

Since Ω is assumed to be invertible, we see from the definition for hypergeometric data that $y^0_{k,q}(t) - \frac{t^q}{q!} \in e^t \mathcal{R}_T[[e^t]]$ and in particular $y^0_{k,1}(t) - t \in e^t \mathcal{R}_T[[e^t]]$, which implies ${y^0_{k,1}}'(t) - 1 \in e^t \mathcal{R}_T[[e^t]]$. Thus $f = \ln({y^0_{k,1}}'(t)) \in e^t \mathcal{R}_T[[e^t]]$ and we may use Lemma 5.4 to find a mirror transformation ν such that

$$\begin{aligned} HG[\mathcal{I}(\nu(\rho(P_k)))](t) &= e^{-f} HG[\mathcal{I}(\rho(P_k))](t) \\ &= \Omega \left[p^{k+1} + \sum_{r=1}^{k+1} \frac{{y^r_{k,1}}'(t)}{{y^0_{k,1}}'(t)} p^{k+1-r} \right] + \Omega p^k \sum_{q=2}^{\infty} \sum_{r=0}^{q+k} (-1)^{q+1} \frac{{y^r_{k,q}}'(t)}{{y^0_{k,1}}'(t)} p^{q-r} \hbar^{-q+1}. \end{aligned}$$

The previous observation also implies that $\frac{-{y^r_{k,1}}'(t)}{{y^0_{k,1}}'(t)} \in e^t \mathcal{R}_T[[e^t]]$ for all $r = 1 ... k+1$. Because $k > 0$, we may use Lemma 5.5 (repeatedly) to find yet another mirror transformation η such that

$$
\begin{aligned}
HG[\mathcal{I}(\eta(\nu(\rho(P_k))))](t) &= HG[\mathcal{I}(\nu(\rho(P_k)))](t) + \sum_{r=1}^{k+1} \frac{-y^{r}_{k,1}{}'(t)}{y^{0}_{k,1}{}'(t)} HG[\mathcal{I}(P_{k+1-r})](t) \\
&= \Omega p^{k+1} + \Omega p^{k} \sum_{q=2}^{\infty} \sum_{r=0}^{q+k} (-1)^{q+1} \frac{y^{r}_{k,q}{}'(t)}{{}^{0}_{k,1}{}'(t)} p^{q-r} \hbar^{-q+1} \\
&\quad + \sum_{r=1}^{k+1} \Omega p^{k+1-r} \frac{-y^{r}_{k,1}{}'(t)}{y^{0}_{k,1}{}'(t)} \sum_{q=1}^{\infty} \sum_{s=0}^{q+k+1-r} (-1)^{q} y^{s}_{k+1-r,q}(t) p^{q-s} \hbar^{-q} \\
&= \Omega p^{k+1} \left[1 + \sum_{q=1}^{\infty} \sum_{r=0}^{q+k+1} (-1)^{q} y^{r}_{k+1,q}(t) p^{q-r} \hbar^{-q} \right]
\end{aligned}
$$

for some functions $y^{r}_{k+1,q}(t)$. It is clear that $y^{r}_{k+1,q}(t)$ is symmetric of degree r in $\{\lambda\}$, and that $y^{0}_{k+1,q}(t)$ is given by

$$
y^{0}_{k+1,q}(t) = \frac{y^{0}_{k,q+1}{}'(t)}{y^{0}_{k,1}{}'(t)}.
$$

Thus we define P_{k+1} to be $\eta(\nu(\rho(P_k)))$ and the theorem is proved. ∎

6 Mirror Theorems

In this section we combine the results of previous sections to establish mirror theorems. These results express the Euler data arising from a concavex bundle in terms of linear classes on N_d in the cases of one and two marked points. We first show that the $(0,1)$-Euler data induced by a concavex bundle V and the extension constructed in Theorem 5.6 are the same. The case of two markings is established by gluing together the result of the $(0,1)$ case and the analog for $(1,0)$-data, which will be addressed briefly.

6.1 One-point Mirror Theorem

Recall that $\hat{Q}$ denotes the specific (u,v)-Euler data arising from a concavex bundle over $\mathbb{P}^n$ as in Lemma 4.1, and that $I_d : \mathbb{P}^n \hookrightarrow N_d$ is an equivariant embedding of projective spaces defined in (16). The mirror theorem for the one marked point case will follow easily from the next lemma, which explicitly identifies $I_d^*(\hat{Q}_{\vec{k}})$. The lemma will also be useful later for proving recovery lemmas to explicitly calculate Gromov-Witten invariants of interest.

Lemma 6.1 *Let $\phi^d = \prod_{l=0}^n \prod_{m=1}^d (p - \lambda_l - m\hbar) \in H_T^*(\mathbb{P}^n)$ denote the denominator of the coefficient of e^{dt} in the hypergeometric series $HG[\mathcal{I}(\hat{Q}_{\vec{k}})](t)$. Then:*

1. *If π_j, ev_j are as before then*

$$I_d^*(\hat{Q}_{d,\vec{k}}) = (-1)^v \phi^d p^{|\vec{k}''|} \hbar^{v-1} ev_{v+1*} \left(\frac{\pi_{v+1}^*(\mathrm{Euler}_T(V_d) \prod_{j=1}^v ev_j^* p^{k_j})}{\hbar + c_1^T(\mathcal{L}_{d,v+1})} \right)$$

2. *$\hat{Q}$ satisfies $deg_\hbar I_d^*(\hat{Q}_{d,\vec{k}}) \leq (n+1)d + v - 2$ for every $\vec{k}$.*

Proof To prove the first part of the lemma, we show that both sides of the equation have equal localizations at $q_i \in \mathbb{P}^n$ for all $0 \leq i \leq n$. Since $I_d(q_i) = p_{i,0} \in N_d$, we have

$$i_{q_i}^*(I_d^*(\hat{Q}_{d,\vec{k}})) = i_{p_{i,0}}^*(\hat{Q}_{d,\vec{k}}) = \hat{Q}_{d,\vec{k}}(\lambda_i).$$

From the proof of Lemma 4.1, we know

$$\hat{Q}_{d,\vec{k}}(\lambda_i) = (-1)^v i_{p_{i,0}}^*(\phi_{i,0}) \hbar^{v-1} \prod_{j=1}^u \lambda_i^{k_{j+v}} \sum_{\{F_d^{v+1}\}} \int_{(F_d^{v+1})_T} \frac{\pi_{v+1}^*(\mathrm{Euler}_T(V_d) \prod_{j=1}^v ev_j^* p^{k_j})}{\mathrm{Euler}_T(N(F_d^{v+1}))(\hbar + c_1^T(\mathcal{L}_{d,v+1}))}$$

It follows from the definition of ϕ^d and $\phi_{q_i} = \prod_{j \neq i}(p - \lambda_j)$ that

$$i_{p_{i,0}}^*(\phi_{p_{i,0}}) = i_{q_i}^*(\phi_{q_i}) i_{q_i}^*(\phi^d),$$

which, when combined with the above expression for $\hat{Q}_{d,k}(\lambda_i)$, gives

$$\hat{Q}_{d,\mathbf{k}}(\lambda_i) = (-1)^v i_{q_i}^*(\phi_{q_i}) i_{q_i}^*(\phi^d) \hbar^{v-1} \prod_{j=1}^u \lambda_i^{k_{j+v}} \sum_{\{F_d^{v+1}\}} \int_{(F_d^{v+1})_T} \frac{\pi_{v+1}^*(\mathrm{Euler}_T(V_d) \prod_{j=1}^v ev_j^* p^{k_j})}{\mathrm{Euler}_T(N(F_d^{v+1}))(\hbar + c_1^T(\mathcal{L}_{d,v+1}))},$$

where the summation is over fixed components $F_d^{v+1} \subset \overline{M_{0,v+1}}(\mathbb{P}^n, d)$ such that the final marked point is mapped to $q_i \in \mathbb{P}^n$.

On the other hand, if we let

$$\alpha_d = \frac{\pi_{v+1}^*(\mathrm{Euler}_T(V_d) \prod_{j=1}^v p^{k_j})}{\hbar + c_1^T(\mathcal{L}_{d,v+1})},$$

then

$$\begin{aligned} i_{q_i}^*((-1)^v\hbar^{v-1}p^{|\vec{k}''|}\phi^d ev_{v+1*}(\alpha_d)) &= (-1)^v\hbar^{v-1}\int_{(\mathbb{P}^n)_T}\phi_{q_i}p^{|\vec{k}''|}\phi^d ev_{v+1*}(\alpha_d) \\ &= (-1)^v\hbar^{v-1}\int_{(\overline{M_{0,v+1}}(\mathbb{P}^n,d))_T} ev_{v+1}^*(p^{|\vec{k}''|}\phi_{q_i}\phi^d)\alpha_d \\ &= (-1)^v\hbar^{v-1}i_{q_i}^*(\phi_{q_i})i_{q_i}^*(\phi^d)\prod_{j=1}^{u}\lambda_i^{k_{j+v}}\int_{(\overline{M_{0,v+1}}(\mathbb{P}^n,d))_T}\alpha_d. \end{aligned}$$

The last line follows since $ev_{v+1}^*\phi_{q_i}$ vanishes unless the image of that last point is q_i. Hence, in calculating this expression via localization we only need consider fixed components of $\overline{M_{0,v+1}}(\mathbb{P}^n,d)$ such that the last marked point is mapped to q_i. Such components are precisely the ones considered in the proof of Lemma 4.1 to obtain the expression for $\hat{Q}_{d,\vec{k}}(\lambda_i)$. Thus both sides have equal localizations at each fixed point; the first statement in the lemma follows.

To show the degree bound in the second statement of the lemma, we know that $\hat{Q}_{d,\vec{k}}$, and hence $I_d^*(\hat{Q}_{d,\vec{k}})$, are both a priori polynomials in $\hbar$ by virtue of $\hat{Q}$ being (u,v)-Euler data. It is clear that $\deg_\hbar\phi^d = (n+1)d$, so part two of the lemma follows quickly from the first statement and these observations. ∎

As an immediate application, we have the following theorem:

Theorem 6.2 (Mirror Theorem I) *The $(0,1,p,\Omega)$-Euler data $\hat{Q}$ induced by concavex bundle $V \to \mathbb{P}^n$ from Lemma 4.1 and the controlled extension $\hat{P}$ of its height 0 data $\hat{P}_0(=\hat{Q}_0)$ to $(0,1,p,\Omega)$-Euler data given by Theorem 5.6 are equal.*

Notice that we are again using the simplified notation $\vec{k}=k$ for $(0,1)$-Euler data.

Proof We confirm that the two conditions necessary for uniqueness in Lemma 4.3 are met: $\hat{Q}$ and $\hat{P}$ have the same height 0 data by assumption. The height 0 data $\hat{Q}_0$, viewed as $(0,0)$-Euler data, satisfies the hypotheses of Theorem 5.6 by the second part of Lemma 6.1. For $k>0$ the extension of Theorem 5.6 satisfies

$$HG[\mathcal{I}(\hat{P}_k)](t) \equiv \Omega p^k \mod \hbar^{-1}.$$

The second statement in Lemma 6.1 also implies that

$$HG[\mathcal{I}(\hat{Q}_k)](t) \equiv \Omega p^k \mod \hbar^{-1}.$$

The degree bound in Lemma 4.3 is equivalent to requiring that the hypergeometric series agree modulo $\hbar^{-1}$, so the theorem follows. ∎

The entire development of $(0,1)$-Euler data can also be carried out for $(1,0)$-Euler data with a few modifications. In particular, replacing the map $I_d : \mathbb{P}^n \hookrightarrow N_d$ with $I'_d : \mathbb{P}^n \hookrightarrow N_d$, given by

$$I'_d([a_0, ..., a_n]) = [a_0 w_0^d, ..., a_n w_0^d]$$

in all statements concerning $(0,1)$-Euler data will then yield the analogous result for $(1,0)$-Euler data. This leads to a one-point mirror theorem concerning $(1,0)$-Euler data induced by a concavex bundle.

6.2 Two-point Mirror Theorem

Explicitly identifying the $(1,1)$-Euler data induced by concavex $V \to \mathbb{P}^n$ in terms of linear classes on N_d is accomplished by gluing together the results for $(0,1)$ and $(1,0)$-Euler data provided by Theorem 6.2 and its immediate analog for $(1,0)$-data.

Theorem 6.3 (Mirror Theorem II) *The* $(1,1)$*-Euler data* $\hat{Q}$ *induced by a concavex bundle* V *is expressible in terms of linear classes by gluing together its* $(0,1)$*-Euler data, identified by Theorem 6.2 in terms of linear classes, and the analog for* $(1,0)$*-Euler data as prescribed by the Euler condition (9). As a consequence, the associated hypergeometric data at height* $\vec{k} = (k_1, k_2)$ *is given by*

$$HG[\mathcal{I}(\hat{Q}_{\vec{k}})](t) = p^{k_1} HG[\mathcal{I}(\hat{Q}_{(0,k_2)})](t)$$

Proof An equivariant class on N_d is fully determined by its values at fixed points. The Euler condition (9) therefore identifies $(1,1)$-Euler data in terms of its component $(1,0)$-Euler data and $(0,1)$-Euler data. These have been given by Theorem 6.2 in terms of linear classes. The second statement follows from the fact that $I_d^*(Q_k) = p^k I_d^*(Q_0)$ for any $(1,0)$-Euler data Q. ∎

7 Recovery Lemmas

Recall that the Gromov-Witten invariants associated to the concavex bundle V on $\mathbb{P}^n$ are defined as

$$K_d(H^{k_1}, ..., H^{k_m}) = \int_{\overline{M_{0,m}}(\mathbb{P}^n,d)} \text{Euler}(V_d) \prod_{j=1}^{m} ev_j^* H^{k_j} \qquad (22)$$

whenever

$$\dim \overline{M_{0,m}}(\mathbb{P}^n, d) = \dim V_d + |\vec{k}|,$$

in which case we say that the integrand is of *critical dimension*. For brevity, in what follows we will abbreviate the notation in (22) to $K_{d,\vec{k}}$, where $\vec{k} = (k_1, ..., k_m)$.

7.1 Recovery for $(0,v)$-Euler data

The next lemma is useful for extracting the invariants $K_{d,\vec{k}}$ from hypergeometric data associated to $(0,v)$-Euler data.

Lemma 7.1 *If $\hat{Q}$ is $(0,v)$-Euler data induced by a concavex bundle V and $\vec{k} = (k_1,...,k_v)$ is such that the integrand in (22) is of critical dimension, then*

$$\lim_{\lambda_i\to 0}\int_{(\mathbb{P}^n)_T} p^s e^{-pt/\hbar}\frac{I_d^*(\hat{Q}_{d,\vec{k}})}{\prod_{l=0}^n\prod_{m=1}^d(p-\lambda_l-m\hbar)} = \left\{\begin{array}{ll} (-1)^v\hbar^{v-3}(2-v-dt)K_{d,\vec{k}} & s=0 \\ (-1)^v\hbar^{v-2}dK_{d,\vec{k}} & s=1 \\ 0 & s>1 \end{array}\right\} \tag{23}$$

Proof Denote the left-hand side of (23) by A_s. By Lemma 6.1,

$$\lim_{\lambda_i\to 0} I_d^*(\hat{Q}_{d,\vec{k}}) = (-1)^v\hbar^{v-1}\prod_{m=1}^d (H-m\hbar)^{n+1} ev_{v+1*}\left(\frac{\pi_{v+1}^*(\mathrm{Euler}(V_d)\prod_{j=1}^v ev_j^*H^{k_j})}{\hbar+c_1(\mathcal{L}_{d,v+1})}\right).$$

Inserting this expression into A_s, we have

$$\begin{aligned} A_s &= (-1)^v\hbar^{v-1}\int_{\mathbb{P}^n} H^s e^{-Ht/\hbar} ev_{v+1*}\left(\frac{\pi_{v+1}^*(\mathrm{Euler}(V_d)\prod_{j=1}^v ev_j^*H^{k_j})}{\hbar+c_1(\mathcal{L}_{d,v+1})}\right) \\ &= (-1)^v\hbar^{v-1}\int_{\overline{M_{0,v+1}}(\mathbb{P}^n,d)} ev_{v+1}^*H^s e^{-ev_{v+1}^*Ht/\hbar}\cdot\frac{\pi_{v+1}^*(\mathrm{Euler}(V_d)\prod_{j=1}^v ev_j^*H^{k_j})}{\hbar+c_1(\mathcal{L}_{d,v+1})} \\ &= (-1)^v\hbar^{v-1}\int_{\overline{M_{0,v}}(\mathbb{P}^n,d)} \mathrm{Euler}(V_d)\prod_{j=1}^v ev_j^*H^{k_j}\pi_{v+1*}\left[\frac{ev_{v+1}^*H^s e^{-ev_{v+1}^*Ht/\hbar}}{\hbar+c_1(\mathcal{L}_{d,v+1})}\right]. \end{aligned}$$

Since $\mathrm{Euler}(V_d)\prod_{j=1}^v ev_j^*H^{k_j}$ has top degree, this integral is just $K_{d,\vec{k}}$ times a scalar factor given by integrating the expression in brackets over a generic fiber of π_{v+1} (which is isomorphic to $\mathbb{P}^1$). Hence,

$$A_s = (-1)^v\hbar^{v-1}K_{d,\vec{k}}\int_{\mathbb{P}^1}\frac{ev_{v+1}^*H^s}{\hbar}\left(\frac{-c_1(\mathcal{L}_{d,2})}{\hbar} - ev_{v+1}^*H\frac{t}{\hbar}\right) \tag{24}$$

The class $c_1(\mathcal{L}_{d,v+1})$ restricted to a fiber is just the cotangent bundle of $\mathbb{P}^1$ with v marked points, while the evaluation map on the fiber is just f which has degree d. By the Gauss-Bonnet theorem, the former class gives a contribution of

$$\int_{\mathbb{P}^1} i^*c_1(\mathcal{L}_{d,v+1}) = -2+v,$$

while the latter contributes

$$\int_{\mathbb{P}^1} i^* ev_{v+1}^* H = d.$$

Inserting these into (24) and considering the various possibilities for s establishes the lemma. ∎

7.2 Recovery for $(1, v)$-Euler data

We will need a recovery lemma for $(1, v)$-Euler data as well in order to extract Gromov-Witten invariants with one descendent class. These concern integrals of the form

$$K_d(H^{k_1}, ..., H^{k_v}, \tau_i(H^{k_{v+1}-i})) = \int_{\overline{M_{0,v+1}}(\mathbb{P}^n,d)} \text{Euler}(V_d) \prod_{j=1}^{v} ev_j^* H^{k_j} \cdot ev_{v+1}^* H^{k_{v+1}-i} \cdot \psi_{v+1}^i, \tag{25}$$

where again the k_i are chosen so the integral is of correct dimension, and ψ_i is the first Chern class of the line bundle given by the cotangent line at the i-th marking. We will abbreviate (25) notationally to $K^i_{d,\vec{k}}$, where $\vec{k} = (k_1, ..., k_v, k_{v+1})$, so that in particular $K^0_{d,\vec{k}} = K_{d,\vec{k}}$.

Lemma 7.2 *Let $\hat{Q}$ be $(1, v)$-Euler data induced by a concavex bundle V on $\mathbb{P}^n$ and fix $\vec{k} = (k_1, ..., k_v, k_{v+1})$ so that (25) is the correct dimension. For $0 \le k^*_{v+1} \le k_{v+1}$ and $\vec{k}^* = (k_1, ..., k_v, k^*_{v+1})$,*

$$\lim_{\lambda_i \to 0} \int_{(\mathbb{P}^n)_T} p^s e^{-pt/\hbar} \frac{I_d^*(\hat{Q}_{d,\vec{k}^*})}{\prod_{l=0}^n \prod_{m=1}^d (p - \lambda_l - m\hbar)} = (-1)^{v+a} \hbar^{v-2-a} \sum_{j=0}^{a} K^j_{d,\vec{k}} \frac{t^{a-j}}{(a-j)!},$$

where $a = k_{v+1} - k^*_{v+1} - s$ for $0 \le s \le k_{v+1} - k^*_{v+1}$, and is 0 otherwise.

Proof We again calculate the integral by pulling back to $\overline{M_{0,v+1}}(\mathbb{P}^n, d)$ and using the expression for $I_d^*(\hat{Q}_{d,\vec{k}^*})$ given in Lemma (6.1):

$$\begin{aligned} &\lim_{\lambda_i \to 0} \int_{(\mathbb{P}^n)_T} p^s e^{-pt/\hbar} \frac{I_d^*(\hat{Q}_{d,\vec{k}^*})}{\prod_{l=0}^n \prod_{m=1}^d (p - \lambda_l - m\hbar)} \\ &\quad = (-1)^v \hbar^{v-1} \int_{\overline{M_{0,v+1}}(\mathbb{P}^n,d)} ev_{v+1}^* H^{s+k^*_{v+1}} e^{-ev_{v+1}^* H t/\hbar} \cdot \frac{\pi_{v+1}^*(\text{Euler}(V_d) \prod_{j=1}^v ev_j^* H^{k_j})}{\hbar + c_1(\mathcal{L}_{d,v+1})} \\ &\quad = (-1)^v \hbar^{v-2-k_{v+1}+k^*_{v+1}+s} \sum_{j=0}^{k_{v+1}-(k^*_{v+1}+s)} (-1)^{k_{v+1}-(k^*_{v+1}+s)} K_{d,\vec{k}}^{k_{v+1}-(k^*_{v+1}+j+s)} \frac{t^j}{j!}, \end{aligned}$$

where we have expanded everything by power series and taken the terms of critical dimension. The lemma then follows. ∎

8 Calculations

8.1 One-Point Candelas Formula

We will prove a one-point generalization for hypersurfaces of the famous Candelas formula, which expresses mirror symmetry for the quintic in the unmarked case.

The $(n+1)p$-Euler data

$$\hat{P}_0 = \prod_{m=0}^{(n+1)d} ((n+1)\kappa - m\hbar) \tag{26}$$

has hypergeometric data given in the non-equivariant limit in power series expansion as

$$\lim_{\lambda \to 0} HG[\mathcal{I}(\hat{P}_0)](t) = (n+1)p \sum_{q=0}^{\infty} (-1)^q f_q(t) p^q \hbar^{-q}.$$

Define $y_{i,q}(t)$ recursively by

$$y_{0,q}(t) = \frac{f_q(t)}{f_0(t)}; \qquad y_{i,q}(t) = \frac{{y_{i-1,q+1}}'(t)}{{y_{i-1,1}}'(t)} \qquad i \geq 1 \tag{27}$$

Let $V = \mathcal{O}(n+1) \to \mathbb{P}^n$ with $n \geq 4$ induce a sequence of obstruction bundles V_d on $\overline{M_{0,1}}(\mathbb{P}^n, d)$. For dimensional reasons, the associated one-point Gromov-Witten invariants of interest are

$$K_d(H^{n-3}) = \int_{\overline{M_{0,1}}(\mathbb{P}^n,d)} \mathrm{Euler}(V_d) ev_1^* H^{n-3}.$$

Lemma 8.1 (One-point Candelas Formula for Hypersurfaces) *Let the Gromov-Witten potential Φ for the degree $n+1$ hypersurface in $\mathbb{P}^n$ be defined as*

$$\Phi(t) = \frac{(n+1)}{2} t^2 + \sum K_d(H^{n-3}) e^{dt}.$$

Then

$$\Phi(T) = (n+1)\left[y_{0,1}(t) y_{n-3,1}(t) - y_{n-3,2}(t)\right],$$

where the $y_{i,q}(t)$ are defined by (27) and the mirror change of variables is

$$T(t) = y_{0,1}(t).$$

Proof Let $\hat{Q}_0$ be the $(0,0)$-Euler data induced by $V = \mathcal{O}(n+1)$ and $\hat{P}_0$ be as in (26). As a consequence of Lemma 3.3 of [LLY97], mirror transformations of $\hat{Q}_0$ and $\hat{P}_0$ agree, which we will denote $\mu(\hat{Q}_0)$ and $\nu(\hat{P}_0)$, in such a way that

$$HG[\mathcal{I}(\mu(\hat{Q}_0))(t) = HG[\mathcal{I}(\hat{Q}_0)(T(t)) = \frac{1}{f_0}HG[\mathcal{I}(\hat{P}_0)(t) = HG[\mathcal{I}(\nu(\hat{P}_0))(t)$$

in the nonequivariant limit. By Theorem 6.2, the extension of $\nu(\hat{P}_0)$ to $(0,1)$-Euler data $\nu(\hat{P})$ given by Theorem 5.6 and the transformed $(0,1)$-Euler data $\mu(\hat{Q})$ agree (here μ now refers to the extension given in Lemma 5.3 of the mirror transformation of [LLY97] to all heights). In particular, at height $n-3$,

$$\begin{aligned} HG[\mathcal{I}(\mu(\hat{Q}_{n-3}))(t) &= HG[\mathcal{I}(\hat{Q}_{n-3})(T) \\ &= \Omega p^{n-3}\left[1 - \frac{1}{n+1}\Phi'(T)\frac{p}{\hbar} + \frac{1}{n+1}(T\Phi'(T) - \Phi(T))\frac{p^2}{\hbar^2}\right] \end{aligned}$$

agrees with

$$HG[\mathcal{I}(\nu(\hat{P}_{n-3})) = \Omega p^{n-3}\left[1 - y_{n-3,1}\frac{p}{\hbar} + y_{n-3,2}\frac{p^2}{\hbar^2}\right].$$

The above expression for $HG[\mathcal{I}(\hat{Q}_{n-3})(T)$ is a direct consequence of Lemma 7.1. The theorem then follows easily by equating terms in these expansions. ∎

8.2 One-Point Closed Formula for $rk(V^-) \geq 2$

Let $V = V^+ \oplus V^-$ be a concavex bundle on $\mathbb{P}^n$ such that the rank of the concave part is at least two and which splits:

$$V = \bigoplus_{l_i > 0}\mathcal{O}(l_i)\bigoplus_{k_i > 0}\mathcal{O}(-k_i),$$

with at least two k_i factors. We also require $\sum l_i + \sum k_i = n+1$, so that V is associated with a (non-compact) Calabi-Yau. When

$$k = n - 2 - rk(V^+) + rk(V^-) > 0 \tag{28}$$

the integrand in

$$K_d(H^k) = \int_{\overline{M_{0,1}}(\mathbb{P}^n,d)} \text{Euler}(V_d)\text{ev}_1^* H^k \tag{29}$$

has the correct dimension.

Lemma 8.2 *The one-point Gromov-Witten invariants $K_d(H^k)$ in 29 with k as in 28 and $rk(V^-) = 2$ are given by*

$$K_d(H^k) = (-1)^{d(\sum k_i)} \frac{\prod l_i \prod_{l_i} (l_i d)! \prod_{k_i} (k_i d - 1)!}{(d!)^{n+1}}$$

and are 0 *if $rk(V^-) > 2$.*

As a special case, when $V = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, we recover the multiple covering formula for one marked point found in the integrality conjectures:

$$K_{d,1} = \frac{1}{d^2}$$

Although this contribution follows easily from the original Aspinwall-Morrison formula and the Divisor Equation [HKK+03], the methods here give another simple proof. Other easy cases of the lemma agree with calculations appearing elsewhere, for instance $V = \mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathbb{P}^2$ was considered in Section 3.2 of [KP08] (where the authors' ultimate interest was evidence for genus one integrality conjectures).

We now prove Lemma 8.2:

Proof The proof is actually just an application of the recover result Lemma 7.1 without use of the mirror theorems for marked points. Let

$$\Omega = \frac{e(V^+)}{e(V^-)} = (-1)^{\sum k_i} p^{(\sum l_i - \sum k_i)} \frac{\prod l_i}{\prod k_i}$$

If $\hat{Q}_0$ is the $(0, 0, p, \Omega)$-Euler data induced by V and $\hat{P}_0$ is the $(0, 0, p, \Omega)$-Euler data given by

$$\hat{P}_d = \prod_{l_i} \prod_{m=0}^{l_i d} (l_i \kappa - m\hbar) \prod_{k_i} \prod_{m=1}^{k_i d - 1} (-k_i \kappa + m\hbar)$$

then by the results of [LLY97], $\hat{Q}_0$ and $\hat{P}_0$ are linked and, because $rk(V^-) \geq 2$, immediately satisfy

$$HG[\mathcal{I}(Q_0)](t) \equiv HG[\mathcal{I}(P_0)](t) \mod \hbar^{-2}.$$

This is sufficient to conclude that $\hat{Q}_0 = \hat{P}_0$ (Lemma 3.1).

Now let $\hat{Q}$ be the $(1, 0, p, \Omega)$-Euler data induced by V (Lemma 4.1). Notice that by Lemma 6.1,

$$I_d^*(\hat{Q}_k) = p^k I_d^*(\hat{Q}_0).$$

By inserting this into Lemma 7.2 in the case of $(1, 0)$-Euler data and using the identification $\hat{Q}_0 = \hat{P}_0$, we find

$$\begin{aligned}
\hbar^{-2}K_d(H^n) &= \lim_{\lambda_i\to 0}\int_{(\mathbb{P}^n)_T} e^{-pt/\hbar}\frac{I_d^*(\hat{Q}_{d,k})}{\prod_{l=0}^n\prod_{m=1}^d(p-\lambda_l-m\hbar)}\\
&= \lim_{\lambda_i\to 0}\int_{(\mathbb{P}^n)_T} e^{-t/\hbar}\frac{p^k I_d^*(\hat{Q}_{d,0})}{\prod_{l=0}^n\prod_{m=1}^d(p-\lambda_l-m\hbar)}\\
&= \lim_{\lambda\to 0}\int_{(\mathbb{P}^n)_T} e^{-pt/\hbar}\frac{p^k\prod_{l_i}\prod_{m=0}^{l_i d}(l_i p-m\hbar)\prod_{k_i}\prod_{m=1}^{k_i d-1}(-k_i p+m\hbar)}{\prod_{l=0}^n\prod_{m=1}^d(p-\lambda_l-m\hbar)}\\
&= (-1)^{d(\sum k_i)}\frac{\prod l_i\prod_{l_i}(l_i d)!\prod_{k_i}(k_i d-1)!}{\hbar^2(d!)^{n+1}}.
\end{aligned}$$

The last line follows by picking off the term of proper degree when $\mathrm{rk}(V^-)=2$. If $\mathrm{rk}(V^-)>2$ then that term is 0 for dimensional reasons. ∎

8.3 Generalized Multiple Covering Formula

Let $V=\mathcal{O}(-1)^{n+1}$ on $\mathbb{P}^n$, which induces on $\overline{M_{0,2}}(\mathbb{P}^n,d)$ a sequence of obstruction bundles V_d of dimension $(n+1)d-n-1$. Since $\overline{M_{0,2}}(\mathbb{P}^n,d)$ has dimension $(n+1)d+n-1$, the only two-point invariants for dimensional reasons are

$$K_d(H^n,H^n)=\int_{\overline{M_{0,2}}(\mathbb{P}^n,d)}\mathrm{Euler}(V_d)ev_1^*H^n ev_2^*H^n.$$

Lemma 8.3

$$K_d(H^n,H^n)=(-1)^{(n+1)(d-1)}\frac{1}{d}$$

This agrees with the calculation made in [LLW07].

Proof The $(-p)^{-n-1}$-Euler data $\hat{P}_0=\prod_{m=1}^{d-1}(-p+m\hbar)^{n+1}$ is immediately equal to the $(-p)^{-n-1}$-Euler data $\hat{Q}_0$ induced by V as they are linked by [LLY97] and their hypergeometric series agree mod $\hbar^{-2}$. The simple extension of $\hat{P}_0$ to $(0,1)$-Euler data of Lemma 5.1 given by $I_d^*\hat{P}_{d,k}=(p-d\hbar)^k I_d^*\hat{P}_{d,0}$ has the property that

$$HG[\mathcal{I}(\hat{P}_n)](t)\equiv\frac{(-1)^{n+1}}{p}\mod \hbar^{-1}.$$

On the other hand, the $(0,1)$-Euler data $\hat{Q}$ induced by V satisfies

$$HG[\mathcal{I}(\hat{Q}_n)](t)\equiv\frac{(-1)^{n+1}}{p}\mod \hbar^{-1}$$

as well by Lemma 6.1. These two $(0,1)$-Euler data then agree by Lemma 4.3. The two-point mirror theorem Theorem 6.3 then yields that the $(1,1)$-Euler data $\hat{Q}$ induced by V satisfies

$$HG[\mathcal{I}(\hat{Q}_{(n,n)})](t) = p^n HG[\mathcal{I}(\hat{P}_n)](t) = p^n HG[\mathcal{I}(\hat{P}_n)](t).$$

One may then calculate

$$\begin{aligned}
-\hbar^{-1}K_d(H^n, H^n) &= \lim_{\lambda_i \to 0} \int_{(\mathbb{P}^n)_T} e^{-pt/\hbar} \frac{I_d^*(\hat{Q}_{d,n,n})}{\prod_{l=0}^n \prod_{m=1}^d (p - \lambda_l - m\hbar)} \\
&= \lim_{\lambda \to 0} \int_{(\mathbb{P}^n)_T} e^{-pt/\hbar} p^n \frac{(p-d\hbar)^n \prod_{m=1}^{d-1}(-p+m\hbar)^{n+1}}{\prod_{l=0}^n \prod_{m=1}^d (p - \lambda_l - m\hbar)} \\
&= (-1)^{(n+1)(d-1)+1} \frac{1}{d\hbar}
\end{aligned}$$

per Lemma 7.2, from which the lemma follows. ∎

8.4 Example of General Case

We now give an example typical of the general case computable by the methods in this paper. Consider the bundle

$$V = \mathcal{O}(3) \oplus \mathcal{O}(-3) \to \mathbb{P}^5.$$

V induces a sequence of obstruction bundles V_d on $\overline{M_{0,1}}(\mathbb{P}^5, d)$ (or $\overline{M_{0,2}}(\mathbb{P}^5, d)$) of rank $6d$. Since $\overline{M_{0,1}}(\mathbb{P}^5, d)$ has dimension $6d+3$ and $\overline{M_{0,1}}(\mathbb{P}^5, d)$ has dimension $6d+4$, the one and two-point invariants of interest are

$$K_d(\tau_i(H^{3-i})) = \int_{\overline{M_{0,1}}(\mathbb{P}^5,d)} \mathrm{Euler}(V_d) ev_1^* H^{3-i} \psi_1^i.$$

for $0 \le i \le 3$ and

$$K_d(H^2, \tau_i(H^{2-i})) = \int_{\overline{M_{0,2}}(\mathbb{P}^5,d)} \mathrm{Euler}(V_d) ev_1^* H^2 ev_2^* H^{2-i} \psi_2^i.$$

for $0 \le i \le 2$. The other invariants computable by the methods in this paper follow quickly from the divisor property or symmetry and are not included.

Let $\hat{Q} = \{\hat{Q}_{d,k}\}$ be the $(0, 1, p, -1)$-Euler data induced by V as in Lemma 4.1. Define

$$g(t) = \sum_{d \ge 1} \frac{(-1)^d (3d)!^2}{d(d!)^6} e^{dt} \in e^t \mathcal{R}_T[[e^t]];$$

By Lemma 5.3 there exists a mirror transformation μ such that for every height k,

$$HG[\mathcal{I}(\mu(\hat{Q})_k)](t) = HG[\mathcal{I}(\hat{Q}_k)](t + g(t)). \tag{30}$$

In particular, at height $k = 0$ (corresponding to the $(0,0)$-Euler data induced by V) the associated hypergeometric data has the expansion

$$\begin{aligned} HG[\mathcal{I}(\mu(\hat{Q})_0)](t) &= HG[\mathcal{I}(\hat{Q}_0)](t+g(t)) \\ &= -1\left[1 - \frac{p}{\hbar}(t+g(t))\right] + O(\hbar^{-2}). \end{aligned}$$

Here we are using using Lemma 6.1 to calculate the expansion.

Now consider the $(0,0)$-Euler data $\hat{P}_0$ defined by

$$\hat{P}_{d,0} = \prod_{m=0}^{3d}(3\kappa - m\hbar)\prod_{m=1}^{3d-1}(-3\kappa + m\hbar).$$

$\hat{P}_0$ is linked to $\hat{Q}_0$ (and hence $\mu(\hat{Q})_0$) by [LLY97]. The hypergeometric data for $\hat{P}_0$ has series expansion of the form

$$\begin{aligned} HG[\mathcal{I}(\hat{P}_0)](t) &= \Omega \cdot \sum_{q=0}^{\infty}\sum_{r=0}^{q}(-1)^q y^r_{0,q}(t)p^{q-r}\hbar^{-q} \\ &= -1\left[1 - \hbar^{-1}(p \cdot y^0_{0,1}(t) + y^1_{0,1}(t))\right] + O(\hbar^{-2}) \end{aligned}$$

Let $f(t) = y^1_{0,1}(t) \in e^t\mathcal{R}_T[[e^t]]$. By Lemma 5.4, there exists a mirror transformation ν so that

$$\begin{aligned} HG[\mathcal{I}(\nu(\hat{P}_0))](t) &= e^{f/\hbar}HG[\mathcal{I}(\hat{P}_0)](t) \\ &= -1\left[1 - \frac{p}{\hbar} \cdot y^0_{0,1}(t)\right] + O(\hbar^{-2}). \end{aligned}$$

One may explicitly check that $y^0_{0,1} = g(t) + t$, so that

$$HG[\mathcal{I}(\mu(\hat{Q})_0)](t+g(t)) \equiv HG[\mathcal{I}(\nu(\hat{P}_0))](t) \mod \hbar^{-2},$$

which, by Lemma 3.1, implies that $\mu(\hat{Q})_0 = \nu(\hat{P}_0)$ as $(0,0)$-Euler data.

By Theorem 6.2, the extension of $\hat{P}_0$ of Theorem 5.6 and the transformed $(0,1)$-Euler data $\mu(\hat{Q})$ induced by V agree. For instance, this implies

$$HG[\mathcal{I}(\hat{Q}_3)](t+g(t)) = \frac{-\hbar}{y^0_{3,3}}\frac{\partial}{\partial t}\left[\frac{-\hbar}{y^0_{2,2}}\frac{\partial}{\partial t}\left[\frac{-\hbar}{y^0_{1,1}}\frac{\partial}{\partial t}HG[\mathcal{I}(\hat{P}_0)](t)\right]\right].$$

from which one can calculate the one-point Gromov-Witten invariants $K_d(H^3)$.

More generally, Theorem 6.3 along with the recovery lemmas 7.1 and 7.2 may be used to recover one and two-point Gromov-Witten with descendents, which are calculated by computer and given in the accompanying tables.

d	$K_d(H^3)$	$\eta_d(H^3)$
1	144	144
2	-15228	-15264
3	3387832	3387816
4	-1033328799	-1033324992
5	$\frac{9395106912144}{25}$	375804276480
6	152957189840958	-152957190686220
7	$\frac{3299075934458784120}{49}$	67328080295077224
8	$\frac{-125586661840964581023}{4}$	-31396665459982813056
9	$\frac{13773818502953069338 1824}{9}$	1530424278105896555488 8
10	$\frac{-1931628807991093301409032 28}{25}$	-772651523196446715670464 0

Table 1: One-point Gromov-Witten invariants

d	$K_d(\tau_1(H^2))$	$K_d(\tau_2(H))$	$K_d(\tau_3(1))$
1	27	207	-414
2	$\frac{136485}{8}$	$\frac{-339471}{16}$	$\frac{38799}{16}$
3	-4712813	$\frac{51696073}{18}$	$\frac{137491061}{108}$
4	$\frac{100423232037}{64}$	$\frac{-150374595087}{256}$	$\frac{-268540355185}{512}$
5	$\frac{-74841919774848}{125}$	$\frac{1483860161171187}{10000}$	$\frac{41652053617157379}{200000}$
6	$\frac{50249578265872917}{200}$	$\frac{-168846749581461909}{4000}$	$\frac{-210023294358530448 53}{240000}$
7	$\frac{-968660782674268810158}{8575}$	$\frac{305923239810305478432 99}{2401000}$	$\frac{130929262370881810353 02337}{336140000}$
8	$\frac{335595165704465492253 17133}{627200}$	$\frac{-678590997243490327156 420149}{175616000}$	$\frac{-447547582585761567954 290766657}{24586240000}$
9	$\frac{-873077405632389938867 605433}{33075}$	$\frac{433871533077860082210 0647627393}{4000752000}$	$\frac{892796890201108228402 28699723304167}{10081895040000}$
10	$\frac{235451197166325440745 0821413}{175}$	$\frac{-519364358435358705731 5746796417}{23520000}$	$\frac{-791809569634596848121 327607679163149}{177811200000}$

Table 2: One-point Gromov-Witten invariants with descendents

d	$K_d(H^2, H^2)$	$\eta_d(H^2, H^2)$
1	261	117
2	$\frac{-141669}{2}$	-70965
3	28141053	28140966
4	$\frac{-52415855109}{4}$	-13103928360
5	$\frac{33295202406036}{5}$	6659040481155
6	$\frac{-7152693165982701}{2}$	-3576346597038222
7	$\frac{13967902006221361284}{7}$	1995414572317337289
8	$\frac{-9158575282812536703813}{8}$	-1144821910345015106088
9	670934720575777355387210	670934720575777346006859
10	$\frac{-1999257192955989705434315922}{5}$	$-399851438591201270607089595$

Table 3: Two-point Gromov-Witten invariants

d	$K_d(H^2, \tau_1(H))$	$K_d(H^2, \tau_2(1))$
1	-117	-27
2	$\frac{-31995}{4}$	$\frac{239355}{8}$
3	6726101	-10610038
4	$\frac{-62339468859}{16}$	$\frac{292304178171}{64}$
5	$\frac{55140229054008}{25}$	$\frac{-1097087319721233}{500}$
6	$\frac{-25282225051035849}{20}$	$\frac{113016673920662199}{100}$
7	$\frac{18050357542984580663 1}{245}$	$\frac{-104711048349983088280 11}{17150}$
8	$\frac{-9769931994721059552908 97}{2240}$	$\frac{4282957569679898268601 3239}{125440}$
9	$\frac{824467132742507663200450 42}{315}$	$\frac{-148158083894051727638751258 73}{75600}$
10	$\frac{-556142823059437637502883909 2}{35}$	$\frac{4499962304088589402305334576101}{39200}$

Table 4: Two-point Gromov-Witten invariants with descendents

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